Euler number of Instanton Moduli space and Seiberg-Witten invariants

A.Sako[†],T.Sasaki[*]

[†]*Department of Mathematics, Hiroshima University, Higashi-Hiroshima 739-8526 ,Japan*
[*]*Department of Physics, Hokkaido University,Sapporo 060-0810,Japan*

ABSTRACT

We show that a partition function of topological twisted $N = 4$ Yang-Mills theory is given by Seiberg-Witten invariants on a Riemannian four manifolds under the condition that the sum of Euler number and signature of the four manifolds vanish. The partition function is the sum of Euler number of instanton moduli space when it is possible to apply the vanishing theorem. And we get a relation of Euler number labeled by the instanton number k with Seiberg-Witten invariants, too. All calculation in this paper is done without assuming duality.

[†]sako@math.sci.hiroshima-u.ac.jp
[*] sasaki@particle.sci.hokudai.ac.jp

1 Introduction

The aim of this paper is to get a relation of the partition function of topological twisted $N = 4$ gauge theory with Seiberg-Witten invariants in four manifolds.
The partition function is given by Euler number of instanton moduli space in some conditions. We will show that the Euler number labeled by instanton number k is expressed by Seiberg-Witten invariants when the sum of Euler number and signature of the base four manifolds vanishes. This result gives us the formulas to get the partition function of the twisted $N = 4$ gauge theory by Seiberg-Witten invariants.

The partition functions of the $N = 4$ Yang-Mills theories on some four manifolds are calculated by Vafa-Witten with topological field theory [1] [2]. It is an $SL(2, Z)$ modular form. $SL(2, Z)$ transformation is understood as an extension of Montonen-Olive duality [3]. So the duality relation is apparent in that partition function.

This duality is deeply connected with the Hilbert scheme picture of instanton moduli space [4]. But, in general, instanton moduli space has variety compactification and the sum of Euler number of any compactified moduli space is not necessarily a modular form. Actually, in our calculus, the partition function is not modular form with no contrivance. On the other side, $N = 4$ gauge theory is given by the toroidal compactification of 10-dim $N = 1$ gauge theory on a 4-dim manifold. (Note that "compactification" is used two ways.) So the theory is interpreted as a low energy theory of the Heterotic or TypeI string theory. Recent developments of string theory show us many evidences of duality relation in field theory. In our case, Vafa shows us one method to link the compactified instanton moduli space with the Hilbert scheme [5]. This fact implies that a choice of compactification is understood in string theory better than field theory. We discuss the problem of compactification and duality later.

For our purpose we use a similar tool to [6]. They used the non-abelian monopole theory and related the Donaldson invariants to Seiberg-Witten invariants without using duality [7] [8]. We also calculate the partition function in low energy limit of cohomological field theory [9] and there is no request of S-duality. This is the most different point from Dijkgraaf-Park-Schroers [10]. They have determined the partition function of $N = 4$ supersymmetric Yang-Mills theory on a Kähler surface, using S-duality. Their result is given by Seiberg-Witten invariants, too. So it is interesting to compare our results with theirs.

What we do first is to extend the instanton moduli space to non-abelian monopole moduli [11] [12]. In usual cohomological field theory, it was done in [11]. Vafa-Witten theory is constructed as a balanced topological field theory (we denote it as BTFT in the following) [13]. BTFT has no ghost number anomaly, and its partition function is a sum of Euler number of given zero-section space under vanishing theorem. In the 2nd

section, we will construct the non-abelian monopole theory as BTFT and investigate some character of the theory. The vanishing theorem is an obstruction to construct the partition function as the sum of Euler number of the monopole moduli, and to get a relation with Vafa-Witten theory. We do not study this case closer in this paper.
In the 3rd section, we get the formulas between the partition function of a twisted $N = 4$ Yang-Mills theory and Seiberg-Witten invariants. To get them, we break the balance of topological charge. The tools in this paper were used in getting a relation of Donaldson invariants and Seiberg-Witten invariants [6]. We use a model which has a gauge multiplet that is balanced and a hypermultiplet that is not balanced. We call the model unbalanced topological QCD. Vacuum expectation value of an observable is calculated and the relation between Euler number of instanton moduli space and Seiberg-Witten invariants is obtained if vanishing theorem is applicable and the sum of Euler number and signature of the four manifolds vanishes. The comparison with [1, 10] is also made in this section. At the last section, we discuss some remaining problems and the possibility of extension.

2 Balanced Topological QCD

In this section, we construct a Balanced Topological QCD (BTQCD), which is a twisted $N = 4$ Yang-Mills theory coupled with massive hypermultiplets in the fundamental representation [10, 11, 6].

2.1 Balanced Topological QCD

Let X be a compact Riemannian four manifold and E be an $SU(2)$-bundle over X. The bundle E is classified by the instanton number

$$k = \frac{1}{8\pi^2}\int_X TrF \wedge F, \tag{2.1}$$

where Tr is the trace in the fundamental representation of $SU(2)$ and $F \in \Omega^2_X(\mathcal{G}_E)$ is the adjoint valued curvature 2-form on X. We denote the group of gauge transformations by $\mathcal{G}$, i.e. elements of $\mathcal{G}$ are sections of P, where P is the associated principal $SU(2)$-bundle over X. We pick a *spin*^{c} structure c on X and consider the associated *spin*^{c} bundle $W_c^{\pm}$. Let $\mathcal{A}$ be the space of all connections on P and $\Gamma(W_c^+ \otimes E)(\Gamma(W_c^- \otimes E))$ the space of the sections of the *spin*^{c} bundle twisted by the vector bundle E. After twisting, the complex boson in the hypermultiplet becomes a section of $\Gamma(W_c^+ \otimes E)(\Gamma(W_c^- \otimes E))$;

$$\begin{aligned} &q \in \Gamma(W_c^+ \otimes E),\ q^\dagger \in \Gamma(\bar{W}_c^+ \otimes \tilde{E}), \\ &B \in \Gamma(W_c^- \otimes E),\ B^\dagger \in \Gamma(\bar{W}_c^- \otimes \tilde{E}), \end{aligned} \tag{2.2}$$

where $\tilde{E}$ denotes the vector bundle conjugate to E. The *spin*^{c} Dirac operator

$$\sigma^\mu D_\mu : \Gamma(W_c^+ \otimes E) \to \Gamma(W_c^- \otimes E), \tag{2.3}$$

is the Dirac operator for the *spin*^{c} bundle twisted by E. We will sometimes denote $\sigma^\mu D_\mu$ by $\not{D}$ or $\not{D}_c^E$.

Throughout this paper, we restrict our attention to the case that the gauge group is $SU(2)$ and the theory is coupled with hypermultiplets in the fundamental representation.

algebra of BTQCD

In this paragraph, the algebra of BTQCD is given.

We introduce two global supercharges $Q_\pm$ carrying an additive quantum number (ghost number) $U = \pm 1$. When they act on fields in the adjoint representation, they satisfy the following commutation relations:

$$Q_+^2 = \delta_\theta^g,\ \{Q_+, Q_-\} = -\delta_c^g,\ Q_-^2 = -\delta_{\bar{\theta}}^g, \tag{2.4}$$

where δ_θ^g denotes the gauge transformation generated by adjoint scalar field $\theta \in \Omega_X^0(\mathcal{G}_E)$ and we adopt $\delta_\theta^g A_\mu = D_\mu \theta,\ \delta_\theta^g B_{+\mu\nu} = i[B_{+\mu\nu}, \theta],\ \delta_\theta^g c = i[c, \theta]$. When they act on fields in the fundamental representation, they satisfy the following commutation relations:

$$Q_+^2 = -\delta_\theta^g,\ \{Q_+, Q_-\} = \delta_c^g,\ Q_-^2 = \delta_{\bar{\theta}}^g, \tag{2.5}$$

where we also introduce $U(1)$ global transformation generated by $m \in iR$ and we adopt $\delta_\theta^g q = (i\theta + m)q,\ \delta_\theta^g q^\dagger = q^\dagger(-i\theta - m),\ \delta_\theta^g B = (i\theta + m)B,\ \delta_\theta^g B^\dagger = B^\dagger(-i\theta - m)$. The relative sign difference between (2.4) and (2.5) is simply the difference of representations. A simple explanation is the following. One can construct a field J^a in the adjoint representation with a pair of fields $q, q^\dagger$ in the fundamental representation,

$$J^a \equiv q^\dagger T^a q. \tag{2.6}$$

Using above transformations, one can check (2.4) follows from (2.5),

$$\begin{aligned} Q_+^2 J^a &= Q_+^2 (q^\dagger T^a q) \\ &= (-\delta_\theta^g q^\dagger) T^a q + q^\dagger T^a (-\delta_\theta^g q) \\ &= i[q^\dagger T q, \theta]^a \\ &= \delta_\theta^g J^a. \end{aligned} \tag{2.7}$$

Note that the relative sign difference between (2.4) and (2.5) is consistent with this derivation. The recipe for giving mass to fields in the fundamental representation by global symmetry is considered by Hyun-Park-Park(H-P-P)[6].

We define $\delta_\pm$ transformations $\delta_\pm \equiv [Q_\pm, *\}$. $\delta_\pm$ transformations are given in Appendix A. See also the references [13, 6].

action of BTQCD

Using above fields and transformations, we define the action of BTQCD as

$$h^2 S = \int \sqrt{g}\mathcal{L}, \tag{2.8}$$

where

$$\mathcal{L} = \delta_+\delta_-\mathcal{F}. \tag{2.9}$$

$\mathcal{F}$ is described with fields in the previous paragraph and has ghost number 0. The general recipe for constructing a balanced topological field theory is given by Moore et.al[13].

$\mathcal{F}$ is explicitly given by,

$$\begin{aligned}\mathcal{F} \;=\; & (B_+^{\mu\nu a}s^a_{+\mu\nu}) - (\chi_+^{I\mu\nu a}\psi^a_{B\mu\nu}) - (\chi^{IIa}_{B\mu}\psi^{\mu a}) + (-i\frac{1}{3}B_+^{\mu\nu a}[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma}) \\ & +(B^{\dagger\alpha}s_\alpha) - (\chi_q^{I\dagger\alpha}\psi_{B\alpha}) - (\chi^{II\dagger}_{B\dot{\alpha}}\psi_q^{\dot{\alpha}}) \\ & +(s^{\dagger\alpha}B_\alpha) + (\psi_B^{\dagger\alpha}\chi^I_{q\alpha}) + (\psi^\dagger_{q\dot{\alpha}}\chi_B^{II\dot{\alpha}}) \\ & +(\xi^a\eta^a),\end{aligned} \tag{2.10}$$

where

$$s_+^{\mu\nu} = F_+^{\mu\nu} + q^\dagger\bar{\sigma}^{\mu\nu}q \tag{2.11}$$

$$s_\alpha = (\not{D}q)_\alpha. \tag{2.12}$$

Finally, full lagrangian is given by

$$\mathcal{L}^{full} = \delta_+\delta_-\mathcal{F}. \tag{2.13}$$

Explicit expression of this lagrangian is given in appendix A. This lagrangian (a.7) is different from [1] in matter fields (q, B etc.) and also different from H-P-P[6] in dual fields ($B_+^{\mu\nu}, c, B$ etc.). But due to its construction, it is balanced.

2.2 Fixed Point

In this subsection, we study the nature of the action given in subsection 2.1. Here in particular we investigate the fixed points and vanishing theorem[1].

Fixed Point

To check the nature of lagrangian, we decompose the bosonic part of lagrangian (a.7)

$$\mathcal{L}^{full}_{boson} = \mathcal{L}^{eq}_{boson} + \mathcal{L}^{pro}_{boson}, \tag{2.14}$$

where

$$
\begin{aligned}
\mathcal{L}^{eq}_{boson} &= -H^{I\mu\nu a}_{+}\{H^{Ia}_{+\mu\nu} - (s^a_{+\mu\nu} - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)\} \\
&\quad -H^{II\rho a}_{B}\{H^{IIa}_{B\rho} - (-2D^\mu B^a_{+\mu\rho} + iB^\dagger \sigma_\rho T^a q - iq^\dagger \bar{\sigma}_\rho T^a B - D_\rho c^a)\} \\
&\quad -H^{I\dagger\alpha}_{q}\{H^{I}_{q\alpha} - (s_\alpha + icB_\alpha + m_c B_\alpha)\} + (h.c.) \\
&\quad -H^{II\dagger}_{B\dot\alpha}\{H^{II\dot\alpha}_{B} - (-(\not{D}B)^{\dot\alpha} + (\bar{\sigma}^{\mu\nu}B_{+\mu\nu}q)^{\dot\alpha} + icq^{\dot\alpha} + m_c q^{\dot\alpha})\} + (h.c.)
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
\mathcal{L}^{pro}_{boson} &= -\{[\theta,\bar\theta]^a[\bar\theta,\theta]^a - [c,\theta]^a[c,\bar\theta]^a + [B^{\mu\nu}_{+},\bar\theta]^a[B_{+\mu\nu},\theta]^a\} + D_\mu\bar\theta^a D^\mu\theta^a \\
&\quad +(-iq^\dagger\bar\theta - q^\dagger\bar m)(i\theta q + mq) + (-iq^\dagger\theta - q^\dagger m)(i\bar\theta q + \bar m q) \\
&\quad +(-iB^\dagger\bar\theta - B^\dagger\bar m)(i\theta B + mB) + (-iB^\dagger\theta - B^\dagger m)(i\bar\theta B + \bar m B).
\end{aligned}
\tag{2.16}
$$

$\mathcal{L}^{eq}_{boson}$ is defining the moduli space that we want to consider and $\mathcal{L}^{pro}_{boson}$ is induced for the projection to gauge normal direction. (2.15) lagrangian is rewritten as

$$
\begin{aligned}
\mathcal{L}^{eq}_{boson} &= H \;\; square \;\; terms \\
&\quad +\frac{1}{4}(s^a_{+\mu\nu} - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma})^2 - \frac{1}{4}([B_{+\mu\nu}, c]^a)^2 \\
&\quad +\frac{1}{4}(-2D^\mu B^a_{+\mu\rho} + iB^\dagger\sigma_\rho T^a q - iq^\dagger\bar\sigma_\rho T^a B)^2 + \frac{1}{4}(D_\rho c^a)^2 \\
&\quad +\frac{1}{2}|s|^2 + \frac{1}{2}|icB + m_c B|^2 \\
&\quad +\frac{1}{2}| - (\not{D}^\dagger B)^{\dot\alpha} + (\bar\sigma^{\mu\nu}B_{+\mu\nu}q)^{\dot\alpha}|^2 + \frac{1}{2}|icq + m_c q|^2.
\end{aligned}
\tag{2.17}
$$

Thus we have the following fixed point equations

$$
\begin{gathered}
F_{+\mu\nu} + q^\dagger\bar\sigma_{\mu\nu}q - i[B_{+\mu\rho}, B_{+\nu\sigma}]g^{\rho\sigma} = 0 \\
-2D_\mu B^{\mu\nu}_{+} + iB^\dagger\sigma^\nu q - iq^\dagger\bar\sigma^\nu B = 0 \\
s = \not{D}q = 0 \\
-\not{D}^\dagger B + \bar\sigma_{\mu\nu}B^{\mu\nu}_{+}q = 0 \\
D_\nu\theta = D_\nu c = D_\nu\bar\theta = 0 \\
[\theta,\bar\theta] = [c,\theta] = [c,\bar\theta] = [B^{\mu\nu}_{+},\theta] = [B^{\mu\nu}_{+},\bar\theta] = [B^{\mu\nu}_{+},c] = 0 \\
(i\theta + m)q = (i\bar\theta + \bar m)q = (ic + m_c)q = 0 \\
q^\dagger(-i\theta - m) = q^\dagger(-i\bar\theta - \bar m) = q^\dagger(-ic - m_c) = 0 \\
(i\theta + m)B = (i\bar\theta + \bar m)B = (ic + m_c)B = 0 \\
B^\dagger(-i\theta - m) = B^\dagger(-i\bar\theta - \bar m) = B^\dagger(-ic - m_c) = 0.
\end{gathered}
\tag{2.18}
$$

If hypermultiplet fields are set to zero ($q = q^\dagger = B = B^\dagger = 0$), then above equations are Vafa-Witten equations [1][10]. Thus we call above equations Extended Vafa-Witten equations.

problem

In the previous paragraph, we have obtained fixed point equations of BTQCD. The equations for fermionic zero-modes are just the linearization of the fixed point equation and the condition that they are orthogonal to gauge orbits. Due to the balanced structure each fermionic zero-mode has a partner with the opposite U-number. Thus there is no ghost-number anomaly and the partition function is well defined, i.e.there is no need to insert observables. We want to compute the partition function of BTQCD. According to Vafa-Witten if an appropriate vanishing theorem holds, the partition function becomes the sum of Euler number of moduli space which we want to calculate. Roughly speaking, vanishing theorem is understood as the condition that dual fields $(B_{+\mu\nu}, c, B, B^{\dagger}$ etc.) are to be zero and the dimensions of their moduli space become zero, when we choose an appropriate metric[1]. But we could not verify that vanishing theorem holds in this model. To compare the result of this section to that of the next section, we give the only result to compute the partition function of BTQCD on the condition that vanishing theorem holds.

2.3 result

In this subsection, we give the result of computing the path integral of BTQCD. We define partition function of BTQCD as

$$Z = \frac{1}{Vol\mathcal{G}(2\pi)^{\Omega}} \int \mathcal{D}W\mathcal{D}\psi_W \mathcal{D}Q^{\dagger}\mathcal{D}\psi_Q^{\dagger}\mathcal{D}Q\mathcal{D}\psi_Q e^{-S}, \tag{2.19}$$

where

$$\begin{aligned} W &= A_{\mu}, B_{+}^{\mu\nu}, H_B^{\mu}, H_{+}^{\mu\nu}, \theta, c, \bar{\theta} \\ \psi_W &= \psi_{\mu}, \psi_B^{\mu\nu}, \chi_B^{II\mu}, \chi_{+}^{I\mu\nu}, \xi, \eta \\ Q &= q, B, H_q^I, H_B^{II} \\ \psi_Q &= \psi_q, \psi_B, \chi_q^I, \chi_B^{II} \\ \Omega &= \dim of\ H's. \end{aligned} \tag{2.20}$$

Here we denote auxiliary fields as $H_B^{\mu}, H_{+}^{\mu\nu}, H_q^I, H_B^{II}$, and we call auxiliary fields for Y as $H's\ of\ Y$ in the following. $\dim of\ H's$ is a number of the auxiliary fields.

After path-integrations of transverse part we get the partition function as the sum of two branches, according to the methods of the next section,

$$Z = Z^{V-W} + Z^{B-U(1)S-W}. \tag{2.21}$$

Z^{V-W} is a contribution from branch 1) (gauge symmetry is unbroken), and corresponds to Vafa-Witten partition function. $Z^{B-U(1)S-W}$ is a contribution from branch 2) (gauge

symmetry is broken to $U(1)$), and corresponds to balanced $U(1)$ monopole theory. The fixed point equations of the balanced $U(1)$ monopole theory are

$$F^3_{+\mu\nu} + \frac{1}{2}{q^\dagger}_1 \bar{\sigma}_{\mu\nu} q_1 = 0$$

$$-2\nabla_\mu B_+^{\mu\nu 3} + i\frac{1}{2}{B^\dagger}_1 \sigma^\nu q_1 - i\frac{1}{2}{q^\dagger}_1 \bar{\sigma}^\nu B_1 = 0$$

$$\not{D}^3 q_1 = 0$$

$$-\not{D}^{\dagger 3} B_1 + \frac{1}{2}\bar{\sigma}_{\mu\nu} B_+^{\mu\nu 3} q_1 = 0. \tag{2.22}$$

Where $F^3_{+\mu\nu}$ is a curvature of $U(1)$ left symmetry after breaking $SU(2)$ and the labels of q_1 are B_1 are the ones of color. Since we do not know the vanishing theorem for dual fields $({B_{+\mu\nu}}^3, B_1, {B^\dagger}_1)$ from (2.22), we stop to investigate this model further more in this paper.

3 Unbalanced Topological QCD

In this section, we compute a correlation function of an appropriate BRS exact operator in Unbalanced Topological QCD. As a result, we can describe Euler number of instanton moduli space with Seiberg-Witten invariants. We have a similar but not the same expression to Dijkgraaf et.al[10], because we treat the different theory from theirs. We discuss this point at the end of this section.

3.1 Unbalanced Topological QCD

Here we construct Unbalanced Topological QCD, which is a twisted $N = 4$ Yang-Mills theory coupled with only one massive hypermultiplet in the fundamental representation (we denote it as UBTQCD in the following). Alternatively one get a UBTQCD, when one set one massive hypermultiplet $(B, \psi_B, \chi_B^{II}, H_B^{II})$ of BTQCD in the previous section to zero (we call this process breaking balanced structure).

algebra of UBTQCD

The algebra of UBTQCD is given as a part of the BTQCD algebra. Contrary to the previous section, we only consider the global supercharge Q_+. When it acts on adjoint (fundamental) fields, it satisfies the following commutation relation:

$$Q_+^2 = \delta^g_\theta(-\delta^g_\theta). \tag{3.1}$$

We adopt the same δ_+ transformations as the previous section and appendix A of (a.2.1)$\sim$(a.3.2).

action of UBTQCD

We define the action of UBTQCD as

$$h^2 S = \int d^4x \sqrt{g}\mathcal{L} \tag{3.2}$$

where

$$\mathcal{L} = \delta_+ \Psi. \tag{3.3}$$

We explicitly give Ψ as;

$$\begin{aligned} \Psi \;=\; & -\chi_+^{I\mu\nu a}\{H_{+\mu\nu}^{Ia} - (s_{+\mu\nu}^a - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)\} \\ & -\chi^{II\rho a}\{H_{B\rho}^{IIa} - (-2D^\mu B_{+\mu\rho}^a - D_\rho c^a)\} \\ & -\chi_q^{I\dagger\alpha}\{H_{q\alpha}^I - s_\alpha\} \\ & -\{H_q^{I\dagger\alpha} - s^{\dagger\alpha}\}\chi_{q\alpha}^I \\ & +\{i[\theta,\bar\theta]^a\eta^a - i\xi^a[c,\bar\theta]^a\} + i[B_+^{\mu\nu},\bar\theta]^a\psi_{B\mu\nu}^a + D_\mu\bar\theta^a\psi^{\mu a} \\ & -(-iq_{\dot\alpha}^\dagger\bar\theta - \bar m q_{\dot\alpha}^\dagger)\psi_q^{\dot\alpha} - \psi_{q\dot\alpha}^\dagger(i\bar\theta q^{\dot\alpha} + \bar m q^{\dot\alpha}), \end{aligned} \tag{3.4}$$

where

$$s_+^{\mu\nu a} = F_+^{\mu\nu a} + q^\dagger\bar\sigma^{\mu\nu}T^a q \tag{3.5}$$

$$s^\alpha = (\not{D} q)^\alpha. \tag{3.6}$$

Finally the full lagrangian is given by

$$\begin{aligned} \mathcal{L}^{full} \;=\; & \delta_+\Psi \qquad\qquad (3.7) \\ =\; & -H_+^{I\mu\nu a}\{H_{+\mu\nu}^{Ia} - (s_{+\mu\nu}^a - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)\} \\ & -\chi_+^{I\mu\nu a}\{-i[\chi_{+\mu\nu}^I,\theta]^a + 2D_\mu\psi_\nu^a + \psi_q^\dagger\bar\sigma_{\mu\nu}T^a q + q^\dagger\bar\sigma_{\mu\nu}T^a\psi_q - 2i[B_{+\mu\rho},\psi_{B\nu\sigma}]^a g^{\rho\sigma} \\ & -i[\psi_{B\mu\nu}, c]^a - i[B_{+\mu\nu},\xi]^a\} \\ & -H_B^{II\rho a}\{H_{B\rho}^{IIa} - (-2D^\mu B_{+\mu\rho}^a - D_\rho c^a)\} \\ & -\chi_B^{II\rho a}\{-i[\chi_{B\rho}^{II},\theta]^a - 2D^\mu\psi_{B\mu\rho}^a - 2i[\psi^\mu, B_{+\mu\rho}]^a - D_\rho\xi^a - i[\psi_\rho, c]^a\} \\ & -H_q^{I\dagger\alpha}\{H_{q\alpha}^I - s_\alpha\} \\ & -\chi_q^{I\dagger}\{\not{D}\psi_q + \sigma_\rho i\psi^\rho q\} \\ & +(h.c.\ \ above\ \ two\ \ lines) \\ & -\{[\theta,\bar\theta]^a[\bar\theta,\theta]^a - [c,\theta]^a[c,\bar\theta]^a + [B_+^{\mu\nu},\bar\theta]^a[B_{+\mu\nu},\theta]^a\} + D_\mu\bar\theta^a D^\mu\theta^a \\ & i[\theta,\eta]^a\eta^a + i\xi^a[\xi,\bar\theta]^a + i\xi^a[c,\eta]^a + i[\psi_B^{\mu\nu},\bar\theta]^a\psi_{B\mu\nu}^a + i[B_+^{\mu\nu},\eta]^a\psi_{B\mu\nu}^a + D_\mu\eta^a\psi^{\mu a} + i[\psi_\mu,\bar\theta]^a\psi^{\mu a} \\ & +(-iq^\dagger\bar\theta - q^\dagger\bar m)(i\theta q + mq) + (-iq^\dagger\theta - q^\dagger m)(i\bar\theta q + \bar m q) \\ & +2\psi_q^\dagger(i\bar\theta + \bar m)\psi_q - 2\chi_q^{I\dagger}(i\theta + m)\chi_q^I - (-iq^\dagger\eta - q^\dagger\eta_m)\psi_q + \psi_q^\dagger(i\eta q + \eta_m q). \end{aligned} \tag{3.8}$$

Notice that lagrangian (3.8) is given by lagrangian (a.7) of previous section if $(B, \psi_B, H_B^{II}, \chi_B^{II})$ is set to zero.

3.2 Fixed Point

In this subsection, we study the nature of the action given in subsection 3.1. Here in particular we investigate the fixed points and some observable to insert.

Fixed Point

To check the nature of lagrangian, we decompose the bosonic part of lagrangian (3.8)

$$\mathcal{L}_{boson}^{full} = \mathcal{L}_{boson}^{eq} + \mathcal{L}_{boson}^{pro}, \tag{3.9}$$

where

$$\begin{aligned}
\mathcal{L}_{boson}^{eq} \;=\;& -H_{+}^{I\mu\nu a}\{H_{+\mu\nu}^{Ia} - (s_{+\mu\nu}^{a} - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)\} \\
& -H_B^{II\rho a}\{H_{B\rho}^{IIa} - (-2D^\mu B_{+\mu\rho}^a - D_\rho c^a)\} \\
& -H_q^{I\dagger\alpha}\{H_{q\alpha}^{I} - s_\alpha\} + (h.c.)
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\mathcal{L}_{boson}^{pro} \;=\;& -\{[\theta,\bar\theta]^a[\bar\theta,\theta]^a - [c,\theta]^a[c,\bar\theta]^a + [B_{+}^{\mu\nu},\bar\theta]^a[B_{+\mu\nu},\theta]^a\} + D_\mu\bar\theta^a D^\mu\theta^a \\
& +(-iq^\dagger\bar\theta - q^\dagger\bar m)(i\theta q + mq) + (-iq^\dagger\theta - q^\dagger m)(i\bar\theta q + \bar m q).
\end{aligned} \tag{3.11}$$

$\mathcal{L}_{boson}^{eq}$ is defining the moduli space that we want to consider here and $\mathcal{L}_{boson}^{pro}$ is induced for the projection to gauge normal direction. (3.10) lagrangian is transformed into

$$\begin{aligned}
\mathcal{L}_{boson}^{eq} \;=\;& -\{H_{+\mu\nu}^{Ia} - \frac{1}{2}(s_{+\mu\nu}^{a} - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)\}^2 \\
& -\{H_{B\rho}^{IIa} - \frac{1}{2}(-2D^\mu B_{+\mu\rho}^a - D_\rho c^a)\}^2 \\
& -2|H_{q\alpha}^{I} - \frac{1}{2}s_\alpha|^2 \\
& +\frac{1}{4}(s_{+\mu\nu}^{a} - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)^2 \\
& +\frac{1}{4}(-2D^\mu B_{+\mu\rho}^a - D_\rho c^a)^2 \\
& +\frac{1}{2}|s_\alpha|^2.
\end{aligned} \tag{3.12}$$

Thus we have the following fixed point equations

$$
\begin{gathered}
F_{+\mu\nu} + q^{\dagger}\bar{\sigma}_{\mu\nu}q - i[B_{+\mu\rho}, B_{+\nu\sigma}]g^{\rho\sigma} - i[B_{+\mu\nu}, c] = 0 \\
-2D_{\mu}B_{+}^{\mu\nu} - D^{\nu}c = 0 \\
s = \not{D}q = 0 \\
D_{\nu}\theta = D_{\nu}\bar{\theta} = 0 \\
[\theta, \bar{\theta}] = [c, \theta] = [c, \bar{\theta}] = [B_{+}^{\mu\nu}, \theta] = [B_{+}^{\mu\nu}, \bar{\theta}] = 0 \\
(i\theta + m)q = (i\bar{\theta} + \bar{m})q = 0 \\
q^{\dagger}(-i\theta - m) = q^{\dagger}(-i\bar{\theta} - \bar{m}) = 0.
\end{gathered}
\tag{3.13}
$$

problem

In the previous paragraph, we have obtained the fixed point equations of UBTQCD. In the same way as the previous section, the equations for fermionic zero-modes are just the linearization of the fixed point equations and the conditions that they are orthogonal to gauge orbits. Compared with the previous section, UBTQCD does not have balanced structure. In particular the hypermultiplet does not have balanced structure, while adjoint representation fields still have balanced structure. The partition function of unbalanced theory becomes zero due to its ghost number anomaly when the moduli space dimension of mater field is non-zero. Thus to get an well-defined path integral, we have to insert some observable. One can think an observable

$$
I = \int d^4x(q^{\dagger}(i\theta + m)q + \psi_q^{\dagger}\psi_q). \tag{3.14}
$$

Note that this observable itself is BRS exact, i.e.

$$
I = \delta_{+}\frac{1}{2}\int d^4x(-\psi_q^{\dagger}q + q^{\dagger}\psi_q). \tag{3.15}
$$

Thus the expectation value of I is zero according to Ward-Takahashi identity, and the expectation value of e^I becomes zero when this theory has ghost number anomaly. However as we will see, we obtain non-trivial results.

3.3 branch

In this subsection, we will show that the fixed point equations are decomposed to two branches. We take a similar treatment for (3.13) to [6].

Equations

$$
D_{\nu}\theta = D_{\nu}\bar{\theta} = 0, [\theta, \bar{\theta}] = 0 \tag{3.16}
$$

imply that $\theta, \bar{\theta}$ can be diagonalized in the fixed points. If connection A_μ are irreducible, $\theta, \bar{\theta}$ should be zero (the gauge symmetry is unbroken). If connection A_μ are reducible,

$\theta, \bar{\theta}$ can be non-zero (the gauge symmetry is broken down to $U(1)$). When these solutions are applied to

$$
\begin{aligned}
(i\theta + m)q = (i\bar{\theta} + \bar{m})q = 0 \\
q^\dagger(-i\theta - m) = q^\dagger(-i\bar{\theta} - \bar{m}) = 0,
\end{aligned}
\tag{3.17}
$$

we have two branches;
branch 1) $\theta = \bar{\theta} = 0$ and $q = q^\dagger = 0$
or
branch 2) $\theta = \theta^3 T^3 \neq 0, \bar{\theta} = \bar{\theta}^3 T^3 \neq 0$ and $q \neq 0, q^\dagger \neq 0$.
Note that in the branch 2) we choose unbroken $U(1)$ as T^3 direction without a loss of generality.

branch 1) $\theta = \bar{\theta} = 0$ and $q = q^\dagger = 0$, i.e. the gauge symmetry is unbroken. Remaining fixed point equations are

$$
\begin{aligned}
F_+^{\mu\nu} - i[B_{+\mu\rho}, B_{+\nu\sigma}]g^{\rho\sigma} = 0, -2D_\mu B_+^{\mu\nu} = 0, D_\mu c = 0 \\
[B_+^{\mu\nu}, c] = 0.
\end{aligned}
\tag{3.18}
$$

Here one may apply the same condition as Vafa-Witten[1] to induce the vanishing theorem, and get the moduli space of

$$
F_+^{\mu\nu} = 0. \tag{3.19}
$$

branch 2) $\theta = \theta^3 T^3 \neq 0, \bar{\theta} = \bar{\theta}^3 T^3 \neq 0$ and $q \neq 0, q^\dagger \neq 0$, i.e.the gauge symmetry is broken to $U(1)$. Thus the bundle E splits into line bundles, $E = L \oplus L^{-1}$ with $L \cdot L = -k$. Then equations (3.17) are

$$
\begin{aligned}
(i\theta^3 T^3 + m)q &= \begin{pmatrix} \frac{i}{2}\theta^3 + m & 0 \\ 0 & -\frac{i}{2}\theta^3 + m \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \\
(i\bar{\theta}^3 T^3 + \bar{m})q &= \begin{pmatrix} \frac{i}{2}\bar{\theta}^3 + \bar{m} & 0 \\ 0 & -\frac{i}{2}\bar{\theta}^3 + \bar{m} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \\
q^\dagger(-i\theta^3 T^3 - m) &= \begin{pmatrix} q_1^\dagger & q_2^\dagger \end{pmatrix} \begin{pmatrix} -\frac{i}{2}\theta^3 - m & 0 \\ 0 & \frac{i}{2}\theta^3 + m \end{pmatrix} = 0 \\
q^\dagger(-i\bar{\theta}^3 T^3 - \bar{m}) &= \begin{pmatrix} q_1^\dagger & q_2^\dagger \end{pmatrix} \begin{pmatrix} -\frac{i}{2}\bar{\theta}^3 - \bar{m} & 0 \\ 0 & \frac{i}{2}\bar{\theta}^3 + \bar{m} \end{pmatrix} = 0.
\end{aligned}
\tag{3.20}
$$

Thus the only non-trivial solutions for q are either

$$
q = \begin{pmatrix} q_1 \\ 0 \end{pmatrix}, q^\dagger = \begin{pmatrix} q_1^\dagger & 0 \end{pmatrix} \, and \, \frac{i}{2}\theta^3 + m = \frac{i}{2}\bar{\theta}^3 + \bar{m} = 0 \tag{3.21}
$$

or

$$q=\left(\begin{array}{c}0\\ q_2\end{array}\right), q^\dagger=\left(\begin{array}{cc}0 & q_2^\dagger\end{array}\right)\, and\ -\frac{i}{2}\theta^3+m=-\frac{i}{2}\bar{\theta}^3+\bar{m}=0. \tag{3.22}$$

Throughout this paper we pick the non-trivial solutions for q as $q_1\neq 0\ and\ \theta^3=2im$. In this branch the equations

$$[c,\theta]=[c,\bar{\theta}]=[B_+^{\mu\nu},\theta]=[B_+^{\mu\nu},\bar{\theta}]=0 \tag{3.23}$$

imply that non-zero solutions of $B_+^{\mu\nu}, c$ have the same direction T^3 as θ. Finally we get remaining equations

$$\begin{aligned} F_{+\mu\nu}^3+\frac{1}{2}q_1^\dagger\bar{\sigma}_{\mu\nu}q_1&=0\\ -2\nabla^\mu B_{+\mu\nu}^3=\partial^\mu c^3&=0\\ \sigma^\mu\mathcal{D}_\mu q_1&=0, \end{aligned} \tag{3.24}$$

where ∇^μ is the covariant derivative in respect of Levi-Civita connection of background metric $g^{\mu\nu}$. Here we reinterpret $U(1)\otimes U(1)$ (gauge $U(1)$ and $spin^c$ $U(1)$) as a new $U(1)$ ($spin^{c'}$ $U(1)$), or alternately we redefine $W_c^+\otimes\zeta=W_{c'}^+$ as a different $spin^c$ structure $c'=c+2\zeta$, i.e., $\det(W_c^+\otimes\zeta)=L_c\otimes\zeta^2$. As a result, (3.24) can be interpreted as a perturbed Seiberg-witten monopole equation for the $spin^c$ structure c' as well as H-P-P[6] and B_+, c equations for T^3 direction.

3.4 Gaussian integral

In this subsection we compute the path integral of UBTQCD. According to Appendix, we could evaluate the exact path integral of this theory. In this subsection, we only denote the diagonal part of the big matrix (see Appendix) to read the right contribution easily. As we have already mentioned in subsection 3.2, we have to insert some observable of fundamental fields to get an well-defined path integral. Thus we define expectation value of e^I as

$$< e^I >_{m,c,k}=\frac{1}{Vol\mathcal{G}(2\pi)^\Omega}\int\mathcal{D}W\mathcal{D}\psi_W\mathcal{D}Q^\dagger\mathcal{D}\psi_Q^\dagger\mathcal{D}Q\mathcal{D}\psi_Q e^{-S+I}, \tag{3.25}$$

where

$$\begin{aligned} W&=A_\mu, B_+^{\mu\nu}, H_B^\mu, H_+^{\mu\nu},\theta, c,\bar{\theta}\\ \psi_W&=\psi_\mu,\psi_B^{\mu\nu},\chi_B^{II\mu},\chi_+^{I\mu\nu},\xi,\eta\\ Q&=q, H_q^I\\ \psi_Q&=\psi_q,\chi_q^I\\ I&=\int d^4x(q^\dagger(i\theta+m)q+\psi_q^\dagger\psi_q)\\ \Omega&=\dim of\ H's. \end{aligned} \tag{3.26}$$

In a general computation of the path integral of TFT, it is sufficient to keep only quadratic terms for the transverse degrees and compute the one-loop approximations which give a result exactly [9]. Now let us see that in each branch, what is transverse degrees of freedom. Picking a Riemannian metric g , we rescale $g \to tg$ and take $t \to \infty$ limit. In branch 1), the gauge symmetry is unbroken and the matter fields decouple as the transverse degrees of freedom. In branch 2), the gauge symmetry is broken down to $U(1)$ and the hypermultiplet reduces to one of its color. The suppressed color degrees of freedom for hypermultiplet and the components of the $N = 4$ vector multiplet which do not belong to the Cartan subalgebra part become the transverse degrees of freedom.

On the other hand, the path integrals for the non-transverse degrees should be computed exactly. These path integrals correspond to the path integral of Vafa-Witten theory in branch 1) and the path integral of $U(1)$ monopole theory and $U(1)B_+, c$ theory in branch 2).

We will use the notation $< O >_{m,c,k}$ for the VEV evaluated in the massive UBTQCD for a given $spin^c$ and instanton number k.

result of branch 1)

In this branch, the degrees of freedom for the hypermultiplet become the transverse degrees of freedom. One can decompose the lagrangian (3.8) into two parts

$$\mathcal{L} \approx \mathcal{L}^{V-W}(1) + \mathcal{L}^t(1), \tag{3.27}$$

where the Vafa-Witten part

$$\begin{aligned} \mathcal{L}^{V-W}(1) \;=\; & -H_+^{I\mu\nu a}\{H_{+\mu\nu}^{Ia} - (F_{+\mu\nu}^a - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)\} \\ & -\chi_+^{I\mu\nu a}\{-i[\chi_{+\mu\nu}^I, \theta]^a + 2D_\mu\psi_\nu^a - 2i[B_{+\mu\rho}, \psi_{B\nu\sigma}]^a g^{\rho\sigma} - i[\psi_{B\mu\nu}, c]^a - i[B_{+\mu\nu}, \xi]^a\} \\ & -H_B^{II\rho a}\{H_{B\rho}^{IIa} - (-2D^\mu B_{+\mu\rho}^a - D_\rho c^a)\} \\ & -\chi_B^{II\rho a}\{-i[\chi_{B\rho}^{II}, \theta]^a - 2D^\mu\psi_{B\mu\rho}^a - 2i[\psi^\mu, B_{+\mu\rho}]^a - D_\rho\xi^a - i[\psi_\rho, c]^a\} \\ & -\{[\theta,\bar\theta]^a[\bar\theta,\theta]^a - [c,\theta]^a[c,\bar\theta]^a + [B_+^{\mu\nu},\bar\theta]^a[B_{+\mu\nu},\theta]^a\} + D_\mu\bar\theta^a D^\mu\theta^a \\ & +i[\theta,\eta]^a\eta^a + i\xi^a[\xi,\bar\theta]^a + i\xi^a[c,\eta]^a + i[\psi_B^{\mu\nu},\bar\theta]^a\psi_{B\mu\nu}^a \\ & +i[B_+^{\mu\nu},\eta]^a\psi_{B\mu\nu}^a + D_\mu\eta^a\psi^{\mu a} + i[\psi_\mu,\bar\theta]^a\psi^{\mu a} \end{aligned} \tag{3.28}$$

and a quadratic lagrangian due to the transverse degrees

$$\begin{aligned} \mathcal{L}^t(1) \;=\; & -H_q^{I\dagger\alpha}\{H_{q\alpha}^I - s_\alpha\} \\ & -\chi_q^{I\dagger}\not{D}\psi_q \\ & +(h.c.\;\; above\;\; two\;\; lines) \end{aligned}$$

$$
\begin{aligned}
& -2q^\dagger \bar{m} m q + 2\psi_q^\dagger \bar{m} m \psi_q - 2\chi_q^{I\dagger} m \chi_q^I \\
= \; & -2|H_q^I + \cdots|^2 - 2m|\chi_q^I + \cdots|^2 \\
& -\frac{1}{2} q^\dagger (\not{D}^\dagger \not{D} + 4m\bar{m}) q + \frac{1}{2m} \psi_q^\dagger (\not{D}^\dagger \not{D} + 4m\bar{m}) \psi_q \qquad (3.29)
\end{aligned}
$$

One can rewrite the path integral (3.25) in this branch as

$$
< e^I >_{m,c,k} (1) \; = \; \underbrace{\frac{1}{Vol\mathcal{G}(2\pi)^{\Omega'}} \int \mathcal{D}W \mathcal{D}\psi_W e^{-S^{V-W}(1)}}_{\equiv Z^{V-W}_{m,c,k}(1)} \cdot \underbrace{\frac{1}{(2\pi)^{\Omega''}} \int \mathcal{D}Q^\dagger \mathcal{D}\psi_Q^\dagger \mathcal{D}Q \mathcal{D}\psi_Q e^{-S^t(1)+I(1)}}_{\equiv Z^t_{m,c,k}(1)}, \tag{3.30}
$$

where

$$
\begin{aligned}
h^2 S^{V-W}(1) &= \int d^4x \sqrt{g} \mathcal{L}^{V-W}(1) \\
h^2 S^t(1) &= \int d^4x \sqrt{g} \mathcal{L}^t(1) \\
I(1) &= \int d^4x \sqrt{g} (q^\dagger m q + \psi_q^\dagger \psi_q) \\
\Omega' &= \dim of \; adjoint \; H's \\
\Omega'' &= \dim of \; fundamental \; H's.
\end{aligned} \tag{3.31}
$$

For the Vafa-Witten part $Z^{V-W}_{m,c,k}(1)$, we completely follow Vafa-Witten[1]. Thus we have

$$
Z^{V-W}_{m,c,k}(1) \doteq \chi_k, \tag{3.32}
$$

where χ_k stands for the Euler number of instanton moduli space with instanton number k and $\doteq$ means equality under keeping the vanishing theorem as shown in Vafa-Witten. Note that the existence of the vanishing theorem in the previous section is unknown, but, in this case, we have some examples to which we apply the vanishing theorem [1]. When the vanishing theorem is not applicable, we denote this part as $Z^{V-W}_{m,c,k}(1)$ itself. We discuss the problem of compactification of moduli space later.

For the transverse part $Z^t_{m,c,k}(1)$, we first perform H_q^I, χ_q^I integral and get

$$
\frac{1}{(2\pi)^{\Omega''}} \frac{[\det(-2m)]_{(\chi_q^{I\dagger}, \chi_q^I)}}{[\det(-\frac{1}{\pi})]_{(H_q^{I\dagger}, H_q^I)}} = [\det(\frac{m}{2\pi})]_{\Gamma^-} = (\frac{m}{2\pi})^{dim(\Gamma^-_{\lambda>0} \oplus Ker(\not{D}^\dagger))}. \tag{3.33}
$$

Second we perform q, ψ_q integral for zero and non-zero mode respectively and get

$$
\frac{[\det(-\frac{\not{D}^2}{2m})]_{(\psi_q^\dagger, \psi_q)_{non\ 0}}}{[\det(-\frac{\not{D}^2}{4\pi})]_{(q^\dagger, q)_{non\ 0}}} \cdot \frac{[\det(-1)]_{(\psi_q^\dagger, \psi_q)_0}}{[\det(-\frac{m}{2\pi})]_{(q^\dagger, q)_0}} = (\frac{2\pi}{m})^{dim(\Gamma^+_{\lambda>0} \oplus Ker(\not{D}))}. \tag{3.34}
$$

Note that this expression is not exact, but is sufficient to get the right contribution (see Appendix).

Collecting (3.33) and (3.34), one can get

$$Z^{t}_{m,c,k}(1) = (\frac{m}{2\pi})^{dim(\Gamma^{-}_{\lambda>0}\oplus Ker(\not{D}^{\dagger}))}(\frac{2\pi}{m})^{dim(\Gamma^{+}_{\lambda>0}\oplus Ker(\not{D})} = (\frac{2\pi}{m})^{index(\not{D}^{E}_{c})}. \tag{3.35}$$

Finally for $< e^{\hat{v}} >_{m,c,k} (1)$, one can get

$$< e^{\hat{v}} >_{m,c,k} (1) = Z^{V-W}_{m,c,k}(1)\cdot Z^{t}_{m,c,k}(1) = Z^{V-W}_{m,c,k}(1)\cdot(\frac{2\pi}{m})^{index(\not{D}^{E}_{c})} \doteq \chi_k \cdot (\frac{2\pi}{m})^{index(\not{D}^{E}_{c})}, \tag{3.36}$$

where $\doteq$ stands for results in the vanishing theorem case.

result of branch 2)

In this branch, the gauge symmetry is broken down to $U(1)$. The components of any field which do not belong to the Cartan subalgebra part become the transverse variables. That is the $\pm$ components of adjoint fields , i.e. $T_{\pm} = T_1 \pm iT_2$. And the components of the hypermultiplet with the suppressed color index become the transverse variable. One can decompose the lagrangian (3.8) into two parts

$$\mathcal{L} \approx \mathcal{L}^{U(1)}(2) + \mathcal{L}^{t}(2), \tag{3.37}$$

where $\mathcal{L}^{U(1)}(2)$ is the lagrangian of $U(1)$ UBTQCD, and $\mathcal{L}^{t}(2)$ is the quadratic lagrangian due to the transverse degrees.

$U(1)$ part $\mathcal{L}^{U(1)}(2)$ can be further decomposed into two parts

$$\mathcal{L}^{U(1)}(2) = \mathcal{L}^{U(1)}_{mono}(2) + \mathcal{L}^{U(1)}_{B_+,c}(2), \tag{3.38}$$

$$\begin{aligned}
\mathcal{L}^{U(1)}_{mono}(2) \;=\; & -H^{I3\mu\nu}_{+}\{H^{I3}_{+\mu\nu} - (F^{3}_{+\mu\nu} + \frac{1}{2}q^{\dagger}_{1}\bar{\sigma}_{\mu\nu}q_1)\} \\
& -\chi^{I3\mu\nu}_{+}\{2\nabla_{\mu}\psi^{3}_{\nu} + \frac{1}{2}\psi^{\dagger}_{q_1}\bar{\sigma}_{\mu\nu}q_1 + \frac{1}{2}{q^{\dagger}}_{1}\bar{\sigma}_{\mu\nu}\psi_{q_1}\} \\
& -H^{I\dagger}_{q\ 1}\{H^{I}_{q\ 1} - \not{D}q_1\} \\
& -\chi^{I\dagger}_{q\ 1}\not{D}\psi_{q_1} \\
& +(h.c.\ \ above\ \ two\ \ lines) \\
& +\partial_{\mu}\bar{\theta}^{3}\partial^{\mu}\theta^{3} + \partial_{\mu}\eta^{3}\psi^{3\mu} \\
& +2(-i\frac{1}{2}{q^{\dagger}}_{1}\bar{\theta}^{3} - {q^{\dagger}}_{1}\bar{m})(i\frac{1}{2}\theta^{3}q_1 + mq_1) \\
& +2\psi^{\dagger}_{q_1}(i\frac{1}{2}\bar{\theta}^{3} + \bar{m})\psi_{q_1} - 2\chi^{I\dagger}_{q\ 1}(i\frac{1}{2}\theta^{3} + m)\chi^{I}_{q_1} \\
& -(-i\frac{1}{2}{q^{\dagger}}_{1}\eta^{3} - {q^{\dagger}}_{1}\eta_m)\psi_{q_1} + \psi^{\dagger}_{q_1}(i\frac{1}{2}\eta^{3}q_1 + \eta_m q_1),
\end{aligned} \tag{3.39}$$

and

$$\mathcal{L}^{U(1)}_{B_+,c}(2) = -H^{II3\rho}_B\{H^{II3}_{B\rho} - (-2\nabla^\mu B^3_{+\mu\rho} - \partial_\rho c^3)\} - \chi^{II3\rho}_B\{-2\nabla^\mu\psi^3_{B\mu\rho} - \partial_\rho\xi^3\}, \quad (3.40)$$

where the first part $\mathcal{L}^{U(1)}_{mono}(2)$ is $U(1)$ monopole theory, and the second part $\mathcal{L}^{U(1)}_{B_+,c}(2)$ is $U(1)$ B_+, c theory.

The quadratic lagrangian due to the transverse degrees $\mathcal{L}^t(2)$ is

$$\begin{aligned}
\mathcal{L}^t(2) \;=\; & -4|H^{I+}_{+\mu\nu} + \cdots|^2 - 8m|\chi^{I+}_{+\mu\nu} + \cdots|^2 + 16m^2|\bar\theta^+ + \cdots|^2 - 8m|\eta^+ + \cdots|^2 \\
& -A^+_\mu\{(D^{3+*}D^{3+})^{\mu\nu} + (D^3D^{3*})^{\mu\nu} - \tilde B^{3\mu\rho}_+\tilde B^{3\nu}_{+\ \rho} - \frac{1}{2}\tilde q^\dagger_1\bar\sigma^\mu\sigma^\nu\tilde q_1 + (-(\tilde c^3)^2 + 16m\bar m)g^{\mu\nu}\}A^-_\nu \\
& +\frac{1}{2m}\psi^+_\mu\{(D^{3+*}D^{3+})^{\mu\nu} + (D^3D^{3*})^{\mu\nu} - \tilde B^{3\mu\rho}_+\tilde B^{3\nu}_{+\ \rho} - \frac{1}{2}\tilde q^\dagger_1\bar\sigma^\mu\sigma^\nu\tilde q_1 + (-(\tilde c^3)^2 + 16m\bar m)g^{\mu\nu}\}\psi^-_\nu \\
& -4|H^{II+}_B + \cdots|^2 - 8m|\chi^{II+}_B + \cdots|^2 \\
& -B_{+\mu\nu}\{(D^{3+}D^{3+*})^{\mu\rho}g^{\nu\sigma} - 4\tilde B^{3\mu\nu}_+\tilde B^{3\rho\sigma}_+ + (-(\tilde c^3)^2 + 16m\bar m)g^{\mu\rho}g^{\nu\sigma}\}B_{+\rho\sigma} \\
& +\frac{1}{2m}\psi_{B\mu\nu}\{(D^{3+}D^{3+*})^{\mu\rho}g^{\nu\sigma} - 4\tilde B^{3\mu\nu}_+\tilde B^{3\rho\sigma}_+ + (-(\tilde c^3)^2 + 16m\bar m)g^{\mu\rho}g^{\nu\sigma}\}\psi_{B\rho\sigma} \\
& -c^+\{D^{3*}D^3 - \tilde B^{3\mu\nu}_+\tilde B^3_{+\mu\nu} - (\tilde c^3)^2 - 16m\bar m\}c^- \\
& +\frac{1}{2m}\xi^+\{D^{3*}D^3 - \tilde B^{3\mu\nu}_+\tilde B^3_{+\mu\nu} - (\tilde c^3)^2 - 16m\bar m\}\xi^- \\
& -2|H^I_{q\,2} + \cdots|^2 - 4m|\chi^I_{q_2} + \cdots|^2 \\
& -\frac{1}{2}q^\dagger_2\{\not{D}^{3\dagger}\not{D}^3 - 2\bar\sigma_{\mu\nu}\tilde q_1\tilde q^\dagger_1\bar\sigma^{\mu\nu} + 16m\bar m\}q_2 \\
& +\frac{1}{4m}\psi^\dagger_{q_2}\{\not{D}^{3\dagger}\not{D}^3 + 2\bar\sigma_{\mu\nu}\tilde q_1\tilde q^\dagger_1\bar\sigma^{\mu\nu} + 16m\bar m\}\psi_{q_2} \\
& +(cross\ \ terms). \qquad (3.41)
\end{aligned}$$

One can rewrite the path integral (3.25) in this branch as

$$\begin{aligned}
< e^I >_{m,c,k}(2) \;=\; & \underbrace{\frac{1}{Vol\mathcal{G}^3(2\pi)^{\Omega'}}\int\mathcal{D}W^3\mathcal{D}\psi^3_W\mathcal{D}Q^\dagger_1\mathcal{D}\psi^\dagger_{Q1}\mathcal{D}Q_1\mathcal{D}\psi_{Q1}e^{-S^{U(1)}(2)}}_{\equiv Z^{U(1)}_{m,c,k}(2)} \\
& \cdot\underbrace{\frac{1}{Vol\mathcal{G}^\pm(2\pi)^{\Omega''}}\int\mathcal{D}W^\pm\mathcal{D}\psi^\pm_W\mathcal{D}Q^\dagger_2\mathcal{D}\psi^\dagger_{Q2}\mathcal{D}Q_2\mathcal{D}\psi_{Q2}e^{-S^t(2)+I(2)}}_{\equiv Z^t_{m,c,k}(2)}
\end{aligned} \qquad (3.42)$$

where

$$h^2S^{U(1)}(2) = \int d^4x\sqrt{g}\mathcal{L}^{U(1)}(2)$$

$$h^2 S^t(2) = \int d^4x \sqrt{g} \mathcal{L}^t(2)$$

$$I(2) = \int d^4x \sqrt{g} (q_2^\dagger 2m q_2 + \psi_{q_2}^\dagger \psi_{q2})$$

$$\Omega' = \dim of\ H's\ of\ non-transverse\ degrees$$

$$\Omega'' = \dim of\ H's\ of\ transverse\ degrees. \tag{3.43}$$

For $U(1)$ monopole part, we have

$$Z^{U(1)}_{mono} = \frac{1}{Vol\mathcal{G}^3(2\pi)^{\Omega'''}} \int \mathcal{D}W^3_A \mathcal{D}\psi_{W^3_A} \mathcal{D}Q_1^\dagger \mathcal{D}\psi^\dagger_{Q1} \mathcal{D}Q_1 \mathcal{D}\psi_{Q1} e^{-S^{U(1)}_{mono}(2)}, \tag{3.44}$$

where

$$W^3_A = A^{\mu 3}, H^3_{+\mu\nu}, \theta^3, \bar{\theta}^3, q_1, H^I_{q\,1}$$

$$\psi^3_{W_A} = \psi^{\mu 3}_A, \chi^3_{+\mu\nu}, \eta^3, \psi_{q_1}, \chi^I_{q_1}$$

$$h^2 S^{U(1)}_{mono}(2) = \int d^4x \sqrt{g} \mathcal{L}^{U(1)}_{mono}(2)$$

$$\Omega''' = \dim of\ H's\ of\ U(1)\ S-W\ part. \tag{3.45}$$

For this part we follow H-P-P [6]. In a simple type manifold we only need to consider the zero-dimensional moduli space of the Seiberg-Witten monopoles (we call $\mathcal{M}(x)$). Here we denote *spin*^{c} structure c' that we have already mentioned in subsection 3.3 by $2x$ if c' satisfies the condition of the zero-dimensional moduli space ($\dim \mathcal{M}(c') = \frac{c' \cdot c'}{4} - \frac{2\chi + 3\sigma}{4} = 0$), and we call this *spin*^{c} structure x Seiberg-Witten basic class. The moduli space $\mathcal{M}(x)$ consists of a finite set of points. First for the contributions of the zero-dimensional moduli space $\mathcal{M}(x)$, we have

$$\mathcal{N} n_x, \tag{3.46}$$

where $\mathcal{N}$ is the standard renormalization due to the local operators constructed from metric and depends only on χ and σ [8]. n_x is the sum of the number of points counted with a sign and is called the Seiberg-Witten invariants. For the total contribution to $U(1)$ monopole part (3.44), we have to sum (3.46) with all basic classes x and get

$$Z^{U(1)}_{mono} = \mathcal{N} \sum_x n_x, \tag{3.47}$$

For $U(1)$ B_+, c part we have

$$Z^{U(1)}_{B_+} = \frac{1}{(2\pi)^{\Omega''''}} \int \mathcal{D}W^3_{B_+,c} \mathcal{D}\psi^3_{W_{B_+,c}} e^{-S^1_{B_+,c}(2)}, \tag{3.48}$$

where

$$W^3_{B_+,c} = B_+^{\mu\nu 3}, H^{\mu 3}_B, c^3$$

$$\psi^3_{W_{B_+,c}} = \psi^{\mu\nu 3}_B, \chi^{II\mu 3}_B, \xi^3$$

$$h^2 S^{U(1)}_{B_+,c}(2) = \int d^4x\sqrt{g}\mathcal{L}^{U(1)\ B_+,c}(2) \qquad (3.49)$$

$$\Omega'''' = \dim of\ H's\ of\ U(1)\ B_+, c\ part.$$

$Z^{U(1)}_{B_+}$ is the partition function of the cohomological field theory with the fixed point

$$\nabla^\mu B_{+\mu\nu}{}^3 = 0, \partial_\nu c^3 = 0. \qquad (3.50)$$

This partition function is sum of the ± 1 when there are only isolated solutions as usual. The condition that the $Z^{U(1)}_{B_+}$ is non-zero is that the dimensions of the moduli space of the 0 section defined by (3.50) becomes zero. In fact the virtual dimension of this moduli space is calculated to be

$$\Delta = index(d^{*+} + d) = \frac{1}{2}(\chi + \sigma), \qquad (3.51)$$

where χ and σ are Euler number and signature of X respectively. Thus $\Delta = 0$ is a condition that we get non-trivial results. We discuss this point later.

Finally we get

$$Z^{U(1)}_{m,c,k}(2) = \mathcal{N} Z^{U(1)}_{B_+} \sum_x n_x. \qquad (3.52)$$

Now we evaluate the transverse integral $Z^t_{m,c,k}(2)$. Following H-P-P[6], we choose a unitary gauge in which

$$\theta_\pm = 0, \qquad (3.53)$$

where

$$\theta = \theta^3 T^3 + \theta^+ T^+ + \theta^- T^-. \qquad (3.54)$$

In this gauge θ has values on the maximal torus (Cartan sub-algebra). By following the standard Faddev-Povov gauge fixing procedure, we introduce a new nilpotent BRST operator δ with the algebra

$$\delta\theta_\pm = \pm i C_\pm \theta_3, \delta C_\pm = 0, \delta\theta_3 = 0, \delta\bar{C}_\pm = b_\pm, \delta b_\pm = 0, \qquad (3.55)$$

where $C_\pm$ and $\bar{C}_\pm$ are anti-commuting ghosts and anti-ghosts, respectively, and $b_\pm$ are commuting auxiliary fields. The action for gauge fixing terms reads

$$\begin{aligned} S_{m,gauge}(2) &= \delta\frac{1}{mh^2}\int d^4x\sqrt{g}(\theta_+\bar{C}_- + \bar{C}_+\theta_-) \\ &= \frac{1}{mh^2}\int d^4x\sqrt{g}\{\theta_+ b_- + b_+\theta_- + iC_+\theta_3\bar{C}_- + i\bar{C}_+\theta_3 C_-\} \\ &= \frac{1}{mh^2}\int d^4x\sqrt{g}\{\theta_+ b_- + b_+\theta_- - C_+ 2m\bar{C}_- - \bar{C}_+ 2mC_-\}. \qquad (3.56) \end{aligned}$$

From the second line to the third line, we take weak coupling limit and replace θ^3 with $2im$. Note that this action has ghost number 0.

Now consider the transverse part involving adjoint fields. We perform $b_\pm, C_\pm, \bar{C}_\pm, \bar{\theta}^\pm, \eta^\pm$ integral and get

$$[\det(im)]^{\frac{1}{2}2}_{\Omega^0}[\det(-2)]^{\frac{1}{2}2}_{\Omega^0}[\det(\frac{16m^2}{\pi})]^{-\frac{1}{2}}_{\Omega^0}[\det(-8m)]^{\frac{1}{2}}_{\Omega^0} = [\det(2\pi m)]^{\frac{1}{2}}_{\Omega^0} = (2\pi m)^{\frac{1}{2}dim(\Omega^0_{\lambda>0}\oplus Ker(D^3))}. \tag{3.57}$$

- $H^\pm_+, \chi^\pm_+$ integral

$$[\det(2\pi m)]^{\frac{1}{2}}_{\Omega^{2+}} = (2\pi m)^{\frac{1}{2}dim(\Omega^{2+}_{\lambda>0}\oplus Ker(D^{3+*}))}. \tag{3.58}$$

- $H^\pm_B, \chi^\pm_B$ integral

$$[\det(2\pi m)]^{\frac{1}{2}}_{\Omega^1} = (2\pi m)^{\frac{1}{2}dim(\Omega^1_{\lambda>0}\oplus Ker(D^{3+}+D^{3*}))}. \tag{3.59}$$

- $A^\pm, \psi^\pm$ integral for non-zero mode

$$\frac{[\det(-\frac{D^3D^{3*}+D^{3+*}D^{3+}}{m})]^{\frac{1}{2}}_{\psi^\pm_{non\ 0}}}{[\det(-\frac{D^3D^{3*}+D^{3+*}D^{3+}}{2\pi})]^{\frac{1}{2}}_{A^\pm_{non\ 0}}} = [\det(\frac{2\pi}{m})]^{\frac{1}{2}}_{\Omega^1_{non\ 0}} = (\frac{2\pi}{m})^{\frac{1}{2}dim(\Omega^1_{\lambda>0})}. \tag{3.60}$$

- $B^\pm_+, \psi^\pm_B$ integral for non-zero mode

$$\frac{[\det(-\frac{D^{3+}D^{3+*}}{m})]^{\frac{1}{2}}_{\chi^\pm_{B non\ 0}}}{[\det(-\frac{D^{3+}D^{3+*}}{2\pi})]^{\frac{1}{2}}_{B^\pm_{+ non\ 0}}} = [\det(\frac{2\pi}{m})]^{\frac{1}{2}}_{\Omega^{2+}_{non\ 0}} = (\frac{2\pi}{m})^{\frac{1}{2}dim(\Omega^{2+}_{\lambda>0})} \tag{3.61}$$

- $c^\pm, \xi^\pm$ integral for non-zero mode

$$\frac{[\det(-\frac{D^{3*}D^3}{m})]^{\frac{1}{2}}_{\xi^\pm_{non\ 0}}}{[\det(-\frac{D^{3*}D^3}{2\pi})]^{\frac{1}{2}}_{c^\pm_{non\ 0}}} = [\det(\frac{2\pi}{m})]^{\frac{1}{2}}_{\Omega^0_{non\ 0}} = (\frac{2\pi}{m})^{\frac{1}{2}dim(\Omega^0_{\lambda>0})}. \tag{3.62}$$

Now we collect all the contributions of the adjoint transverse part and get

$$\begin{aligned}\frac{1}{(2\pi)^{\Omega''_{ado}}}(2\pi m)^{\frac{1}{2}dim(\Omega^0_{\lambda>0}\oplus Ker(D^3))}(2\pi m)^{\frac{1}{2}dim(\Omega^{2+}_{\lambda>0}\oplus Ker(D^{3+*}))}(2\pi m)^{\frac{1}{2}dim(\Omega^1_{\lambda>0}\oplus Ker(D^{3+}+D^{3*}))}&\\ \cdot(\frac{2\pi}{m})^{\frac{1}{2}dim(\Omega^1_{\lambda>0})}(\frac{2\pi}{m})^{\frac{1}{2}dim(\Omega^{2+}_{\lambda>0})}(\frac{2\pi}{m})^{\frac{1}{2}dim(\Omega^0_{\lambda>0})} &= 1\end{aligned} \tag{3.63}$$

Remaining transverse integral is fundamental part. First we perform $H^I_{q\,2}, \chi^I_{q2}$ integral and get

$$\frac{1}{(2\pi)^{\Omega''_{fun}}}\frac{[\det(-4m)]_{(\chi^\dagger_{q2},\chi_{q2})}}{[\det(-\frac{1}{\pi})]_{(H^\dagger_{q2},H^2_q)}} = [\det(\frac{m}{\pi})]_{\Gamma^-} = (\frac{m}{\pi})^{dim(\Gamma^-_{\lambda>0}\oplus Ker((\not{D}^3)^\dagger))}. \tag{3.64}$$

Next we perform q_2, ψ_{q2} integral for non-zero and zero mode respectively and get

$$\frac{[\det(-\frac{\not{D}^\dagger\not{D}}{4m})]_{(\psi^\dagger_{q2},\psi_{q2})_{non\ 0}}}{[\det(-\frac{\not{D}^\dagger\not{D}}{4\pi})]_{(q^\dagger_2,q_2)_{non\ 0}}}\frac{[\det(-1)]_{(\psi^\dagger_{q2},\psi_{q2})_0}}{[\det(-\frac{m}{\pi})]_{(q^\dagger_2,q_2)_0}} = [\det(\frac{\pi}{m})]_{\Gamma^+} = (\frac{\pi}{m})^{dim(\Gamma^+_{\lambda>0}\oplus Ker((\not{D}^3)))}. \tag{3.65}$$

Collecting (3.64) and (3.65), one can get

$$(\frac{m}{\pi})^{dim(\Gamma^-_{\lambda>0}\oplus Ker((\not{D}^3)^\dagger))}(\frac{\pi}{m})^{dim(\Gamma^+_{\lambda>0}\oplus Ker((\not{D}^3)))} = (\frac{\pi}{m})^{index(\not{D}^3)} \tag{3.66}$$

From (3.63) and (3.66) we get

$$Z^t_{m,c,k}(2) = (\frac{\pi}{m})^{index(\not{D}^3)}. \tag{3.67}$$

Finally for $< e^I >_{m,c,k} (2)$ we get

$$< e^I >_{m,c,k} (2) = Z^{U(1)}_{m,c,k}(2)\cdot Z^2_{m,c,k}(2) = \mathcal{N}(\frac{\pi}{m})^{index(\not{D}^3)} Z^{U(1)}_{B_+}\sum_x n_x. \tag{3.68}$$

synthesis

As we have already mentioned , $< e^I >_{m,c,k}$ itself is zero. However from above two paragraphs each branch has non-trivial contributions. Thus we have finally

$$\begin{aligned} 0 &= Z^{V-W}_{m,c,k}\cdot\left(\frac{2\pi}{m}\right)^{index\ \not{D}^E_c} + \mathcal{N}Z_{B_+}\sum_x n_x\cdot\left(\frac{\pi}{m}\right)^{index\ \not{D}^3} \\ &\doteq \chi_k\cdot\left(\frac{2\pi}{m}\right)^{index\ \not{D}^E_c} + \mathcal{N}Z_{B_+}\sum_x n_x\cdot\left(\frac{\pi}{m}\right)^{index\ \not{D}^3}, \end{aligned} \tag{3.69}$$

where the last expression is valid in the vanishing theorem case.

In general $index\not{D}^E_c$ is calculated to be

$$index\not{D}^E_c = -k + \frac{rank(E)}{8}(c\cdot c - \sigma). \tag{3.70}$$

In this case,

$$c\cdot\zeta = -index\not{D}^E_c + 2\Delta. \tag{3.71}$$

The Dirac operator $\not{D}^3$ which operate on q_2, ψ_{q_2}, and so on is necessary to be understood as the Dirac operator with the connection given by $c - 2\zeta$. Then

$$index\not{D}_c^3 = 0 + \frac{1}{8}((c-2\zeta)\cdot(c-2\zeta)-\sigma) \tag{3.72}$$

$$= \frac{1}{8}(c\cdot c - 4k + 4index\not{D}_c^E - 8\Delta - \sigma). \tag{3.73}$$

Thus we get a relation

$$index\not{D}^3 = index\not{D}_c^E - \Delta \tag{3.74}$$

Inserting (3.74) into (3.69), since m is a free parameter, we get non-trivial result only in the case $\Delta = 0$. Remember that Δ is also the dimension of the moduli space of $U(1)$ B_+, c theory. Thus the condition $\Delta = 0$ is consistent with defining $Z_{B_+}^{U(1)}$ (3.48). $\Delta = 0$ is also consistent with geographic condition, for example simple type condition($b_2^+ > 1$), Furuta theory($b_2 \geq \frac{5}{4}|\sigma| + 2$) and $\frac{11}{8}$ conjecture($b_2 \geq \frac{11}{8}|\sigma|$) [14] [15].

Finally under the condition $\Delta = 0$, from (3.69) we have

$$\chi_k \doteq Z_{m,c,k}^{V-W} = -\mathcal{N} Z_{B_+} \sum_x n_x \left(\frac{1}{2}\right)^{index\not{D}_c^E}. \tag{3.75}$$

Note that above x satisfies that $x \cdot x = \frac{2\chi+3\sigma}{4}$ and $x = \frac{c+2\zeta}{2}$.

We think the Vafa-Witten partition as the sum of (3.75) with weight $e^{\tau k}$, where τ is a parameter. But the sum of this partition function don not clarify modular invariance since $\Delta = 0$ is special case which do not depend on the coupling τ in topological twisted model [1] . Additionally we do not assume duality, then there is no guarantee that our partition function has modular invariance and is same as Vafa-Witten's. We suppose that the difference come from compactification of the moduli space. We do not use the duality relation and our model is not asymptotic-free theory. So, there is possibility that compactification in our theory is not the same as the one in Hilbert scheme. Thus we can describe the twisted $N = 4$ Yang-Mills partition function that may not be the same as Vafa-Witten's partition function with Seiberg-Witten invariants. Our expression is similar to Dijkgraaf[10]. The most significant difference is τ dependence. Dijkgraaf's is τ dependent, while ours is τ independent. The reason why Dijkgraaf's partition function depends on τ is that they treat the physical $N = 4$ Yang-Mills theory itself. According to Labastida[13], the $N = 4$ Yang-Mills theory depends on τ. On the other hand we treat UBTQCD, which is the twisted $N = 4$ Yang-Mills theory coupled with a fundamental hypermultiplet. As we mention above, this difference may cause breaking the modular invariance. In other words, our theory is not conformal invariant, and τ is not possible to be a good parameter. But our computation is done without assumption like duality relation. If there is difference we have to interpret the origin of the difference occurred from compactification [16].

4 Conclusion

We have studied the balanced topological QCD and its broken balance theory and got relations of the partition function of twisted $N = 4$ SU(2) Yang-Mills theory with the partition function of twisted abelian QCD. This relation is understood in several ways. For example, the sum of Euler number of instanton moduli space , which is invariant under $SL(2, Z)$ transformation, is described by Seiberg-Witten invariants when $\Delta = 0$ and the vanishing theorem is valid. In other cases there is no vanishing theorem like §5.4 in [1] , we got a similar but not the same formulas under the condition of $\Delta = 0$. There is no other reasons to understand the difference from the result of Vafa-Witten and Dijkgraaf et al.[1] [10] than the difference of compactification.

Some problems are left for our future work. When $\Delta \neq 0$, can we obtain any similar non-trivial results without assumption of duality relation? We may obtain them by simple reformation. But it is difficult to expect that the partition function have the nature of modular invariance in naive reformation. We are interested in a connection with the duality and a compactification. How can we obtain the modular invariant partition function with no assumption of duality? We have some hints of this question but no answer.
As we saw in section 2, vanishing theorem of BTQCD is not studied in this paper. If the theorem exists, we get the sum of Euler number of non-abelian monopole moduli space as the partition function of the BTQCD. It is an interesting work to investigate the nature of the partition function because the theory has the branches that contain both Vafa-Witten theory and Seiberg-Witten theory.

Acknowledgment

We are grateful to H.Kanno for helpful suggestions and observations and a critical reading of the manuscript. We also would like to thank M.Furuta and K.Ono for valuable discussion. A.S. is supported by JSPS Research Fellowships for Young Scientists.

A the BRS algebra and the BTQCD action

We give the BRS algebra and the lagrangian of BTQCD explicitly in this appendix.

A.1 Algebra

$\delta_\pm$ transformations are given as follows.

$$\begin{cases} \delta_- A_\mu = \chi^{II}_\mu \\ \delta_- \chi^{II}_\mu = -\delta^{\bar{\theta}}_g A_\mu = -D_\mu \bar{\theta} \\ \delta_- \psi_\mu = -\delta^c_g A_\mu - H^{II}_{B\mu} = -D_\mu c - H^{II}_{B\mu} \\ \delta_- H^{II}_{B\mu} = -\delta^c_g \chi^{II}_{B\mu} + \delta_+ \delta^{\bar{\theta}}_g A_\mu = -i[\chi^{II}_{B\mu}, c] + \delta_+ \delta^{\bar{\theta}}_g A_\mu \end{cases} \tag{a.1.1}$$

$$\begin{cases} \delta_- B^{\mu\nu}_+ = \chi^{I\mu\nu}_+ \\ \delta_- \chi^{I\mu\nu}_+ = -\delta^{\bar{\theta}}_g B^{\mu\nu}_+ = -i[B^{\mu\nu}_+, \bar{\theta}] \\ \delta_- \psi^{\mu\nu}_B = -\delta^c_g B^{\mu\nu}_+ - H^{I\mu\nu}_+ = -i[B^{\mu\nu}_+, c] - H^{I\mu\nu}_+ \\ \delta_- H^{I\mu\nu} = -\delta^c_g \chi^{I\mu\nu}_+ + \delta_+ \delta^{\bar{\theta}}_g B^{\mu\nu}_+ = -i[\chi^{I\mu\nu}_+, c] + \delta_+ \delta^{\bar{\theta}}_g B^{\mu\nu}_+ \end{cases} \tag{a.1.2}$$

$$\begin{cases} \delta_- q^{\dot{\alpha}} = \chi^{II\dot{\alpha}}_B \\ \delta_- \chi^{II\dot{\alpha}}_B = \delta^{\bar{\theta}}_g q^{\dot{\alpha}} = i\bar{\theta} q^{\dot{\alpha}} + \bar{m} q^{\dot{\alpha}} \\ \delta_- \psi^{\dot{\alpha}}_q = \delta^c_g q^{\dot{\alpha}} - H^{II\dot{\alpha}}_B = icq^{\dot{\alpha}} + m_c q^{\dot{\alpha}} - H^{II\dot{\alpha}}_B \\ \delta_- H^{II\dot{\alpha}}_B = \delta^c_g \chi^{II\dot{\alpha}}_B - \delta_+ \delta^{\bar{\theta}}_g q^{\dot{\alpha}} = ic\chi^{II\dot{\alpha}}_B + m_c - \delta_+ \delta^{\bar{\theta}}_g q^{\dot{\alpha}} \end{cases} \tag{a.1.3}$$

$$\begin{cases} \delta_- B_\alpha = \chi^I_{q\alpha} \\ \delta_- \chi^I_{q\alpha} = \delta^{\bar{\theta}}_g B_\alpha = i\bar{\theta} B_\alpha + \bar{m} B_\alpha \\ \delta_- \psi_{B\alpha} = \delta^c_g B_\alpha - H^I_{q\alpha} = icB_\alpha + m_c B_\alpha - H^I_{q\alpha} \\ \delta_- H^I_{q\alpha} = \delta^c_g \chi^I_{q\alpha} - \delta_+ \delta^{\bar{\theta}}_g B_\alpha = ic\chi^I_{q\alpha} + m_c \chi^I_{q\alpha} - \delta_+ \delta^{\bar{\theta}}_g B_\alpha \end{cases} \tag{a.1.4}$$

$$\begin{cases} \delta_- q^\dagger_{\dot{\alpha}} = \chi^{II\dagger}_{B\dot{\alpha}} \\ \delta_- \chi^{II\dagger}_{B\dot{\alpha}} = \delta^{\bar{\theta}}_g q^\dagger_{\dot{\alpha}} = -iq^\dagger_{\dot{\alpha}} \bar{\theta} - \bar{m} q^\dagger_{\dot{\alpha}} \\ \delta_- \psi^\dagger_{q\dot{\alpha}} = \delta^c_g q^\dagger_{\dot{\alpha}} - H^{II\dagger}_{B\dot{\alpha}} = -iq^\dagger_{\dot{\alpha}} c - m_c q^\dagger_{\dot{\alpha}} - H^{II\dagger}_{B\dot{\alpha}} \\ \delta_- H^{II\dagger}_{B\dot{\alpha}} = \delta^c_g \chi^{II\dagger}_{B\dot{\alpha}} - \delta_+ \delta^{\bar{\theta}}_g q^\dagger_{\dot{\alpha}} = -i\chi^{I\dagger}_{B\dot{\alpha}} c - m_c \chi^{I\dagger}_{B\dot{\alpha}} - \delta_+ \delta^{\bar{\theta}}_g q^\dagger_{\dot{\alpha}} \end{cases} \tag{a.1.5}$$

$$\begin{cases} \delta_- B^{\dagger\alpha} = \chi^{I\dagger\alpha}_q \\ \delta_- \chi^{I\dagger\alpha}_q = \delta^{\bar{\theta}}_g B^{\dagger\alpha} = -iB^{\dagger\alpha} \bar{\theta} - \bar{m} B^{\dagger\alpha} \\ \delta_- \psi^{\dagger\alpha}_B = \delta^c_g B^{\dagger\alpha} - H^{I\dagger\alpha}_q = -iB^{\dagger\alpha} c - m_c B^{\dagger\alpha} - H^{I\dagger\alpha}_q \\ \delta_- H^{I\dagger\alpha}_q = \delta^c_g \chi^{I\dagger\alpha}_q - \delta_+ \delta^{\bar{\theta}}_g B^{\dagger\alpha} = -i\chi^{I\dagger\alpha}_q c - m_c \chi^{I\dagger\alpha}_q - \delta_+ \delta^{\bar{\theta}}_g B^{\dagger\alpha}. \end{cases} \tag{a.1.6}$$

δ_+ transformations are given by

$$\begin{cases} \delta_+ A_\mu = \psi_\mu \\ \delta_+ \psi_\mu = \delta^\theta_g A_\mu = D_\mu \theta \\ \delta_+ \chi^{II}_{B\mu} = H^{II}_{B\mu} \\ \delta_+ H^{II}_{B\mu} = \delta^\theta_g \chi^{II}_{B\mu} = i[\chi^{II}_{B\mu}, \theta] \end{cases} \tag{a.2.1}$$

$$\left\{\begin{array}{c} \delta_+ B_+^{\mu\nu} = \psi_B^{\mu\nu} \\ \delta_+ \psi_B^{\mu\nu} = \delta_g^\theta B_+^{\mu\nu} = i[B_+^{\mu\nu}, \theta] \\ \delta_+ \chi_+^{I\mu\nu} = H_+^{I\mu\nu} \\ \delta_+ H_+^{I\mu\nu} = \delta_g^\theta \chi_+^{I\mu\nu} = i[\chi_+^{I\mu\nu}, \theta] \end{array}\right. \tag{a.2.2}$$

$$\left\{\begin{array}{c} \delta_+ q^{\dot\alpha} = \psi_q^{\dot\alpha} \\ \delta_+ \psi_q^{\dot\alpha} = -\delta_g^\theta q_q^{\dot\alpha} = -(i\theta q_q^{\dot\alpha} + m q_q^{\dot\alpha}) \\ \delta_+ \chi_B^{II\dot\alpha} = H_B^{II\dot\alpha} \\ \delta_+ H_B^{II\dot\alpha} = -\delta_g^\theta \chi_B^{II\dot\alpha} = -(i\theta \chi_B^{II\dot\alpha} + m\chi_B^{II\dot\alpha}) \end{array}\right. \tag{a.2.3}$$

$$\left\{\begin{array}{c} \delta_+ B_\alpha = \psi_{B\alpha} \\ \delta_+ \psi_{B\alpha} = -\delta_g^\theta B_\alpha = -(i\theta B_\alpha + m B_\alpha) \\ \delta_+ \chi_{q\alpha}^I = H_{q\alpha}^I \\ \delta_+ H_{q\alpha}^I = -\delta_g^\theta \chi_{q\alpha}^I = -(i\theta \chi_{q\alpha}^I + m\chi_{q\alpha}^I) \end{array}\right. \tag{a.2.4}$$

$$\left\{\begin{array}{c} \delta_+ q_{\dot\alpha}^\dagger = \psi_{q\dot\alpha}^\dagger \\ \delta_+ \psi_{q\dot\alpha}^\dagger = -\delta_g^\theta q_{\dot\alpha}^\dagger = -(-i q_{\dot\alpha}^\dagger \theta - m q_{\dot\alpha}^\dagger) \\ \delta_+ \chi_{B\dot\alpha}^{II\dagger} = H_{B\dot\alpha}^{II\dagger} \\ \delta_+ H_{B\dot\alpha}^{II\dagger} = -\delta_g^\theta \chi_{B\dot\alpha}^{II\dagger} = -(-i\chi_{B\dot\alpha}^{II\dagger}\theta - m\chi_{B\dot\alpha}^{II\dagger}) \end{array}\right. \tag{a.2.5}$$

$$\left\{\begin{array}{c} \delta_+ B^{\dagger\alpha} = \psi_B^{\dagger\alpha} \\ \delta_+ \psi_B^{\dagger\alpha} = -\delta_g^\theta B^{\dagger\alpha} = -(-iB^{\dagger\alpha}\theta - m B^{\dagger\alpha}) \\ \delta_+ \chi_q^{I\dagger\alpha} = H_q^{I\dagger\alpha} \\ \delta_+ H_q^{I\dagger\alpha} = -\delta_g^\theta \chi_q^{I\dagger\alpha} = -(-i\chi_q^{I\dagger\alpha}\theta - m\chi_q^{I\dagger\alpha}). \end{array}\right. \tag{a.2.6}$$

Transformations for $c, \theta, \bar\theta, m, m_c, \bar m$ are given by

$$\left\{\begin{array}{ll} \delta_+ \theta = 0 & \\ \delta_- \theta = \xi & , \delta_+ \xi = \delta_g^\theta c = i[c, \theta] \\ \delta_+ c = \xi & , \delta_- \xi = \delta_g^{\bar\theta} \theta = i[\theta, \bar\theta] \\ \delta_- c = \eta & , \delta_+ \eta = \delta_g^{\theta} \bar\theta = i[\bar\theta, \theta] \\ \delta_+ \bar\theta = \eta & , \delta_- \eta = \delta_g^{\bar\theta} c = i[c, \bar\theta] \\ \delta_- \bar\theta = 0 & \end{array}\right. \tag{a.3.1}$$

$$
\begin{cases}
\delta_+ m = 0 & \\
\delta_- m = \xi_m & , \delta_+ \xi_m = 0 \\
\delta_+ m_c = \xi_m & , \delta_- \xi_m = 0 \\
\delta_- m_c = \eta_m & , \delta_+ \eta_m = 0 \\
\delta_+ \bar{m} = \eta_m & , \delta_- \eta_m = 0 \\
\delta_- \bar{m} = 0. &
\end{cases}
\tag{a.3.2}
$$

A.2 action of BTQCD

We write down the lagrangian of BTQCD explicitly in this paragraph.
$\delta_- \mathcal{F}$ is given as

$$
\begin{aligned}
\delta_- \mathcal{F} &= \delta_-(B_+^{\mu\nu a} s^a_{+\mu\nu}) - \delta_-(\chi_+^{I\mu\nu a}\psi^a_{B\mu\nu}) - \delta_-(\chi^{IIa}_{B\mu}\psi^{\mu a}) + \delta_-(-i\frac{1}{3}B_+^{\mu\nu a}[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma}) \\
&\quad +\delta_-(B^{\dagger\alpha}s_\alpha) - \delta_-(\chi_q^{I\dagger\alpha}\psi_{B\alpha}) - \delta_-(\chi^{II\dagger}_{B\dot{\alpha}}\psi_q^{\dot{\alpha}}) \\
&\quad +\delta_-(s^{\dagger\alpha}B_\alpha) + \delta_-(\psi_B^{\dagger\alpha}\chi^I_{q\alpha}) + \delta_-(\psi^\dagger_{q\dot{\alpha}}\chi_B^{II\dot{\alpha}}) \\
&\quad +\delta_-(\xi^a\eta^a) \qquad\qquad (a.4) \\
&= -\chi_+^{I\mu\nu a}\{H^{Ia}_{+\mu\nu} - (s^a_{+\mu\nu} - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)\} \\
&\quad -\chi^{II\rho a}\{H^{IIa}_{B\rho} - (-2D^\mu B^a_{+\mu\rho} + iB^\dagger\sigma_\rho T^a q - iq^\dagger\bar{\sigma}_\rho T^a B - D_\rho c^a)\} \\
&\quad -\chi_q^{I\dagger\alpha}\{H^I_{q\alpha} - (s_\alpha + icB_\alpha + m_c B_\alpha)\} \\
&\quad -\chi^{II\dagger}_{B\dot{\alpha}}\{H_B^{II\dot{\alpha}} - (-(\not{D}^\dagger B)^{\dot{\alpha}} + (\bar{\sigma}^{\mu\nu}B_{+\mu\nu}q)^{\dot{\alpha}} + icq^{\dot{\alpha}} + m_c q^{\dot{\alpha}})\} \\
&\quad -\{H_q^{I\dagger\alpha} - (s^{\dagger\alpha} - iB^{\dagger\alpha}c - m_c B^{\dagger\alpha})\}\chi^I_{q\alpha} \\
&\quad -\{H^{II\dagger}_{B\dot{\alpha}} - (-(\not{D}^\dagger B)^\dagger_{\dot{\alpha}} + (q^\dagger B_{+\mu\nu}\bar{\sigma}^{\mu\nu})_{\dot{\alpha}} - iq^\dagger_{\dot{\alpha}}c - m_c q^\dagger_{\dot{\alpha}})\}\chi_B^{II\dot{\alpha}} \\
&\quad +\{i[\theta, \bar{\theta}]^a\eta^a - i\xi^a[c, \bar{\theta}]^a\} + i[B_+^{\mu\nu}, \bar{\theta}]^a\psi^a_{B\mu\nu} + D_\mu\bar{\theta}^a\psi^{\mu a} \\
&\quad -(-iB^{\dagger\alpha}\bar{\theta} - \bar{m}B^{\dagger\alpha})\psi_{B\alpha} - (-iq^\dagger_{\dot{\alpha}}\bar{\theta} - \bar{m}q^\dagger_{\dot{\alpha}})\psi_q^{\dot{\alpha}} \\
&\quad -\psi_B^{\dagger\alpha}(i\bar{\theta}B_\alpha + \bar{m}B_\alpha) - \psi^\dagger_{q\dot{\alpha}}(i\bar{\theta}q^{\dot{\alpha}} + \bar{m}q^{\dot{\alpha}}), \qquad\qquad (a.5)
\end{aligned}
$$

The full lagrangian is given as

$$
\begin{aligned}
\mathcal{L}^{full} &= \delta_+\delta_-\mathcal{F} \qquad\qquad (a.6) \\
&= -H_+^{I\mu\nu a}\{H^{Ia}_{+\mu\nu} - (s^a_{+\mu\nu} - i[B_{+\mu\rho}, B_{+\nu\sigma}]^a g^{\rho\sigma} - i[B_{+\mu\nu}, c]^a)\} \\
&\quad -\chi_+^{I\mu\nu a}\{-i[\chi^I_{+\mu\nu}, \theta]^a + 2D_\mu\psi^a_\nu + \psi_q^\dagger\bar{\sigma}_{\mu\nu}T^a q + q^\dagger\bar{\sigma}_{\mu\nu}T^a\psi_q - 2i[B_{+\mu\rho}, \psi_{B\nu\sigma}]^a g^{\rho\sigma} \\
&\quad -i[\psi_{B\mu\nu}, c]^a - i[B_{+\mu\nu}, \xi]^a\} \\
&\quad -H_B^{II\rho a}\{H^{IIa}_{B\rho} - (-2D^\mu B^a_{+\mu\rho} + iB^\dagger\sigma_\rho T^a q - iq^\dagger\bar{\sigma}_\rho T^a B - D_\rho c^a)\} \\
&\quad -\chi_B^{II\rho a}\{-i[\chi^{II}_{B\rho}, \theta]^a - 2D^\mu\psi^a_{B\mu\rho} - 2i[\psi^\mu, B_{+\mu\rho}]^a \\
&\quad +i\psi_B^\dagger\sigma_\rho T^a q + iB^\dagger\sigma_\rho T^a\psi_q - i\psi_q^\dagger\bar{\sigma}_\rho T^a B - iq^\dagger\bar{\sigma}_\rho T^a\psi_B - D_\rho\xi^a - i[\psi_\rho, c]^a\}
\end{aligned}
$$

$$
\begin{aligned}
&-H_q^{I\dagger\alpha}\{H^I_{q\alpha}-(s_\alpha+icB_\alpha+m_cB_\alpha)\}\\
&-\chi_q^{I\dagger}\{\not{D}\psi_q+\sigma_\rho i\psi^\rho q+i\xi B+ic\psi_B+\xi_m B+m_c\psi_B\}\\
&+(h.c.\ \ above\ \ two\ \ lines)\\
&-H_{B\dot\alpha}^{II\dagger}\{H_B^{II\dot\alpha}-(-(\not{D}^\dagger B)^{\dot\alpha}+(\bar\sigma^{\mu\nu}B_{+\mu\nu}q)^{\dot\alpha}+icq^{\dot\alpha}+m_cq^{\dot\alpha})\}\\
&-\chi_B^{II\dagger}\{-\not{D}^\dagger\psi_B-\bar\sigma_\rho i\psi^\rho B+(\bar\sigma^{\mu\nu}\psi_{B\mu\nu}q)+(\bar\sigma^{\mu\nu}B_{+\mu\nu}\psi_q)+i\xi q+ic\psi_q+\xi_m q+m_c\psi_q\}\\
&+(h.c.\ \ above\ \ two\ \ lines)\\
&-\{[\theta,\bar\theta]^a[\bar\theta,\theta]^a-[c,\theta]^a[c,\bar\theta]^a+[B_+^{\mu\nu},\bar\theta]^a[B_{+\mu\nu},\theta]^a\}+D_\mu\bar\theta^aD^\mu\theta^a\\
&+i[\theta,\eta]^a\eta^a+i\xi^a[\xi,\bar\theta]^a+i\xi^a[c,\eta]^a+i[\psi_B^{\mu\nu},\bar\theta]^a\psi^a_{B\mu\nu}\\
&+i[B_+^{\mu\nu},\eta]^a\psi^a_{B\mu\nu}+D_\mu\eta^a\psi^{\mu a}+i[\psi_\mu,\bar\theta]^a\psi^{\mu a}\\
&+(-iq^\dagger\bar\theta-q^\dagger\bar m)(i\theta q+mq)+(-iq^\dagger\theta-q^\dagger m)(i\bar\theta q+\bar m q)\\
&+2\psi_q^\dagger(i\bar\theta+\bar m)\psi_q-2\chi_q^{I\dagger}(i\theta+m)\chi_q^I-(-iq^\dagger\eta-q^\dagger\eta_m)\psi_q+\psi_q^\dagger(i\eta q+\eta_m q)\\
&+(-iB^\dagger\bar\theta-B^\dagger\bar m)(i\theta B+mB)+(-iB^\dagger\theta-B^\dagger m)(i\bar\theta B+\bar m B)\\
&+2\psi_B^\dagger(i\bar\theta+\bar m)\psi_B-2\chi_B^{II\dagger}(i\theta+m)\chi_B^{II}-(-iB^\dagger\eta-B^\dagger\eta_m)\psi_B+\psi_B^\dagger(i\eta B+\eta_m B).
\end{aligned}
\tag{a.7}
$$

B the path integral of the transverse part

As we have mentioned in the first part of section3.4, the path integral in 3.4 is not exact, but it amounts to the right result that we will derive in this section. In computation, we take the weak coupling limit. When we replace the non-transverse fields with the fixed point values, we denote $Y_{non-trans}$ by $\tilde{Y}_{non-trans}$. In particular the fixed points of $\theta,\bar\theta$ are given as $\theta=\bar\theta=0$ in branch 1) and $\theta^3=2im,\bar\theta=2i\bar m$ in branch 2). We also discuss the different treatment from the path integral in 3.4 at the end of this section. See [17], too.

B.1 branch 1) and its big matrix

In branch 1), the path integral of the transverse part is

$$
Z^t_{m,c,k}(1) \;=\; \frac{1}{(2\pi)^{\Omega''}}\int \mathcal{D}Q^\dagger\mathcal{D}\psi_Q^\dagger\mathcal{D}Q\mathcal{D}\psi_Q e^{-S^t(1)+I(1)} \tag{b.1}
$$

where

$$
\begin{aligned}
h^2S^t(1)&=\int d^4x\sqrt{g}\mathcal{L}^t(1)\\
I(1)&=\int d^4x\sqrt{g}(q^\dagger mq+\psi_q^\dagger\psi_q)\\
\Omega''&=\dim of\ fundamental\ H's.
\end{aligned}
\tag{b.2}
$$

For $\mathcal{L}^t(1)$(3.29), we denote

$$\begin{aligned}\mathcal{L}^t(1) &= -2|H^I_q+\cdots|^2 - 2m|\chi^I_q+\cdots|^2 \\ &\quad -\frac{1}{2}q^\dagger M^b(1)q + \frac{1}{2m}\psi^\dagger_q M^f(1)\psi_q,\end{aligned} \tag{b.3}$$

where

$$M^b(1) = M^f(1) = \not{D}^\dagger\not{D} + 4m\bar{m}. \tag{b.4}$$

In general $M^b(M^f)$ is matrix and is not necessarily diagonalized. M^b and M^f may not be the same as we will see soon. We call $M^b(1)$ big matrix of branch 1).

Before computing (b.1), we briefly review the notion of $index\not{D}$.

One can decompose $q\in\Gamma^+, H^I_q\in\Gamma^-$ into

$$\not{D}^\dagger\not{D}q^\lambda = \lambda q^\lambda$$

$$\not{D}\not{D}^\dagger H^{I\lambda}_q = \lambda H^{I\lambda}_q. \tag{b.5}$$

These decomposition is called spectra decomposition. Note that if $\lambda>0$ then q^λ and $H^{I\lambda}_q$ are isomorphic. However if $\lambda=0$ then q^λ and $H^{I\lambda}_q$ are not isomorphic. $index\not{D}$ measures the difference between $\Gamma^+_{\lambda=0}$ and $\Gamma^-_{\lambda=0}$, and is defined as

$$\begin{aligned}index\not{D} &= \dim\Gamma^+_{\lambda=0} - \dim\Gamma^-_{\lambda=0} \\ &= \dim Ker\not{D} - \dim Ker\not{D}^\dagger,\end{aligned} \tag{b.6}$$

where we denote $\Gamma^+_{\lambda=0} = Ker\not{D}, \Gamma^+_{\lambda=0} = Ker\not{D}^\dagger$.

In computing (b.1), (b.6) emerges when non-kinetic part and off-diagonal part of M are able to be ignored (in this branch simply $m\bar{m}$ terms in (b.4)). This process is achieved by diagonalization and field redefinition. Then we get the expression (b.1) as $index\not{D}$. Conversely it is enough to get this expression that we consider only kinetic diagonal part of M in the path integral.

Now we perform the path integral of the transverse part of branch 1) explicitly.

First for H^I_q, χ^I_q integral,

$$\frac{1}{(2\pi)^{\Omega''}}\frac{[\det(-2m)]_{(\chi^{I\dagger}_q,\chi^I_q)}}{[\det(-\frac{1}{\pi})]_{(H^{I\dagger}_q,H^I_q)}} = [\det(\frac{m}{2\pi})]_{\Gamma^-} = (\frac{m}{2\pi})^{dim(\Gamma^-_{\lambda>0}\oplus Ker(\not{D}^\dagger))}. \tag{b.7}$$

Note that the transformation at the second equality is necessary to derive $index\not{D}$.

For q,ψ_q integral for non-zero mode,

$$\frac{[\det(-\frac{\not{D}^2}{2m})]_{(\psi^\dagger_q,\psi_q)_{non\ 0}}}{[\det(-\frac{\not{D}^2}{4\pi})]_{(q^\dagger,q)_{non\ 0}}} = (\frac{2\pi}{m})^{dim(\Gamma^+_{\lambda>0})}. \tag{b.8}$$

For q, ψ_q integral for zero mode, we consider that the integrant of this path integral comes only from observable $I(1)$ and we get

$$\frac{[\det(-1)]_{(\psi_q^\dagger,\psi_q)_0}}{[\det(-\frac{m}{2\pi})]_{(q^\dagger,q)_0}} = (\frac{2\pi}{m})^{\dim Ker(\not{D})}. \tag{b.9}$$

From (b.8) and (b.9)

$$(\frac{2\pi}{m})^{\dim(\Gamma^+_{\lambda>0}\oplus Ker(\not{D}))}. \tag{b.10}$$

Collecting (b.7) and (b.10),

$$Z^t_{m,c,k} = (\frac{m}{2\pi})^{dim(\Gamma^-_{\lambda>0}\oplus Ker(\not{D}^\dagger))}(\frac{2\pi}{m})^{\dim(\Gamma^+_{\lambda>0}\oplus Ker(\not{D}))} = (\frac{2\pi}{m})^{index\not{D}}. \tag{b.11}$$

Note that $dim(\Gamma^+_{\lambda>0})$ and $dim(\Gamma^-_{\lambda>0})$ cancel each other.

B.2 branch 2) and its big matrix

In branch 2), the path integral of the transverse part is

$$Z^t_{m,c,k}(2) = \frac{1}{Vol\mathcal{G}^\pm(2\pi)^{\Omega''}}\int \mathcal{D}W^\pm\mathcal{D}\psi^\pm_W\mathcal{D}Q_2^\dagger\mathcal{D}\psi^\dagger_{Q2}\mathcal{D}Q_2\mathcal{D}\psi_{Q2}e^{-S^t(2)+I(2)}, \tag{b.12}$$

where

$$h^2S^t(2) = \int d^4x\sqrt{g}\mathcal{L}^t(2)$$
$$I(2) = \int d^4x\sqrt{g}(q_2^\dagger 2mq_2 + \psi^\dagger_{q2}\psi_{q2})$$
$$\Omega'' = \dim of\ H's\ of\ transverse\ degrees. \tag{b.13}$$

For $\mathcal{L}^t(2)$ (3.41), we denote

$$\begin{aligned}\mathcal{L}^t(2) \;=\; & -4|H^{I+}_{+\mu\nu}+\cdots|^2 - 8m|\chi^{I+}_{+\mu\nu}+\cdots|^2 + 16m^2|\bar{\theta}^+ + \cdots|^2 - 8m|\eta^+ + \cdots|^2 \\ & -4|H^{II+}_B+\cdots|^2 - 8m|\chi^{II+}_B+\cdots|^2 \\ & -2|H^I_{q\,2}+\cdots|^2 - 4m|\chi^I_{q2}+\cdots|^2 \\ & -Y^TM^b(2)Y + \frac{1}{2m}\Psi^T_YM^f(2)\Psi_Y,\end{aligned} \tag{b.14}$$

where Y^T, Ψ^T_Y are raw vectors,

$$Y^T = (A^+_\mu, B_{+\nu\rho}{}^+, c^+, q^\dagger{}_2) \tag{b.15}$$

$$\Psi_Y^T = (\psi_\mu^+, \psi_{B\nu\rho}{}^+, \xi^+, \psi_{q_2}^\dagger), \quad \text{(b.16)}$$

and Y, Ψ_Y are column vectors,

$$Y = (A_\sigma^-, B_{+\gamma\delta}{}^-, c^-, q_2) \quad \text{(b.17)}$$

$$\Psi_Y = (\psi_\sigma^-, \psi_{B\gamma\delta}{}^-, \xi^-, \psi_{q_2}). \quad \text{(b.18)}$$

To derive the result (3.67) from (b.12) (b.14), we can neglect the non-kinetic terms and off-diagonal part of $M(2)$ (we will give explicitly later). There is the contribution from Faddeev-Popov determinant of $\theta^\pm = 0$ gauge and it is possible to discard the path integral of $Y^\pm$ for zero mode according to the balanced structure of adjoint fields.

In this remaining subsection, we concentrate on giving $M(2)$ explicitly. $M^b(2)(M^f(2))$ can be decomposed into

$$M^b(2) = \begin{pmatrix} M_{AA}^b & M_{Aq}^b \\ M_{q^\dagger A}^b & M_{q^\dagger q}^b \end{pmatrix}. \quad \text{(b.19)}$$

We denote matrix element of $M^b(2)$ (or $M_{AA}^b, M_{Aq}^b, M_{q^\dagger A}^b, M_{q^\dagger q}^b$) by $\{M^b(2)\}^{A^{+\mu}B_+^{-\gamma\delta}}$, $\{M_{Aq}^b\}^{A^{+\mu}q}$ etc.

diagonal part of $M^b(2)(M^f(2))$

$$\begin{aligned} \{M_{AA}^b\}^{A^{+\mu}A^{-\sigma}} &= \{M_{AA}^f\}^{A^{+\mu}A^{-\sigma}} \\ &= (D^{3+*}D^{3+})^{\mu\sigma} + (D^3D^{3*})^{\mu\sigma} - \tilde{B}_+^{3\mu\rho}\tilde{B}_{+\ \rho}^{3\sigma} - \frac{1}{2}\tilde{q}_1^\dagger\bar{\sigma}^\mu\sigma^\sigma\tilde{q}_1 + (-(\tilde{c}^3)^2 + 16m\bar{m})g^{\mu\sigma} \end{aligned} \quad \text{(b.20)}$$

$$\begin{aligned} \{M_{AA}^b\}^{B_+^{+\nu\rho}B_+^{-\gamma\delta}} &= \{M_{AA}^f\}^{B_+^{+\nu\rho}B_+^{-\gamma\delta}} \\ &= (D^{3+}D^{3+*})^{\nu\rho}g^{\gamma\delta} - 4\tilde{B}_+^{3\nu\rho}\tilde{B}_+^{3\gamma\delta} + (-(\tilde{c}^3)^2 + 16m\bar{m})g^{\nu\rho}g^{\gamma\delta} \end{aligned} \quad \text{(b.21)}$$

$$\begin{aligned} \{M_{AA}^b\}^{c^+c^-} &= \{M_{AA}^f\}^{c^+c^-} \\ &= D^{3*}D^3 - \tilde{B}_+^{3\mu\nu}\tilde{B}_{+\mu\nu}^3 - (\tilde{c}^3)^2 - 16m\bar{m} \end{aligned} \quad \text{(b.22)}$$

$$\{M_{q^\dagger q}^b\}^{q^\dagger q} = \not{D}^{3\dagger}\not{D}^3 - 2\bar{\sigma}_{\mu\nu}\tilde{q}_1\tilde{q}_1^\dagger\bar{\sigma}^{\mu\nu} + 16m\bar{m} \quad \text{(b.23)}$$

$$\{M_{q^\dagger q}^f\}^{q^\dagger q} = \not{D}^{3\dagger}\not{D}^3 + 2\bar{\sigma}_{\mu\nu}\tilde{q}_1\tilde{q}_1^\dagger\bar{\sigma}^{\mu\nu} + 16m\bar{m} \quad \text{(b.24)}$$

Note that $\{M_{q^\dagger q}^b\}^{q^\dagger q}$ and $\{M_{q^\dagger q}^f\}^{q^\dagger q}$ are different.

off-diagonal part of $M^b_{AA}(M^f_{AA})$

$$\begin{aligned}
\{M^b_{AA}\}^{A^{+\mu}B^{-\rho\sigma}_{+}} &= \{M^b_{AA}\}^{A^{+\mu}B^{-\rho\sigma}_{+}} \\
&= i(\overleftarrow{D^{3+}})_\nu \tilde{c}^3 g^{\mu\rho} g^{\nu\sigma} - 2i\tilde{B}^{3\mu\rho}_{+}(D^{3+*})^\rho - i(\overleftarrow{D^{3*}})^\mu \tilde{B}^{3\rho\sigma}_{+} \\
&\quad -2i(\overleftarrow{D^{3+}})_\nu \tilde{B}^{3\mu\rho}_{+} g^{\nu\sigma}
\end{aligned} \tag{b.25}$$

$$\begin{aligned}
\{M^b_{AA}\}^{A^{+\mu}c^{-}} &= \{M^b_{AA}\}^{A^{+\mu}c^{-}} \\
&= -i(\overleftarrow{D^{3+}})_\nu \tilde{B}^{3\mu\nu}_{+} - 2i\tilde{B}^{3\mu\rho}_{+}(D^3)_\rho + i(\overleftarrow{D^{3*}})^\mu \tilde{c}^3
\end{aligned} \tag{b.26}$$

$$\begin{aligned}
\{M^b_{AA}\}^{B^{+\nu\rho}_{+}c^{-}} &= \{M^b_{AA}\}^{B^{+\nu\rho}_{+}c^{-}} \\
&= (\overleftarrow{D^{3+*}})^\nu (D^3)^\rho - 2\tilde{c}^3 \tilde{B}^{3\nu\rho}_{+}
\end{aligned} \tag{b.27}$$

off-diagonal part of $M^b_{Aq}(M^f_{Aq})$

Here using

$$Y^\pm = \frac{1}{2}(Y^1 \mp iY^2), \tag{b.28}$$

we denote $\{M^b_{Aq}\}^{Y^1q}, \{M^b_{Aq}\}^{Y^2q}$ and $\{M^f_{Aq}\}^{Y^1q}, \{M^f_{Aq}\}^{Y^2q}$ instead of $\{M^b_{Aq}\}^{Y^+q}, \{M^f_{Aq}\}^{Y^+q}$. The reason why we cannot denote $\{M^b_{Aq}\}^{Y^+q}, \{M^f_{Aq}\}^{Y^+q}$ is that there are terms $(D^3_\mu A^+_\nu - D^3_\nu A^+_\mu)_+ q^\dagger{}_2 \bar{\sigma}^{\mu\nu}\tilde{q}_1$ and $i(\not{D}^3 q_2)^\dagger \not{A}^- \tilde{q}_1$ exist simultaneously in $\mathcal{L}^t(2)$ (b.14) , for example.

$$\{M^b_{Aq}\}^{A^{1\mu}q} = -\frac{1}{2}(\overleftarrow{D^{3+}})^\nu \tilde{q}^\dagger_1 \bar{\sigma}_{\mu\nu} - i\frac{1}{4}\tilde{q}^\dagger_1 \bar{\sigma}_\mu \not{D}^3 - i\frac{1}{4}(\overleftarrow{D^{3+}})^\mu \tilde{q}^\dagger_1 \tag{b.29}$$

$$\{M^b_{Aq}\}^{A^{2\mu}q} = -i\frac{1}{2}(\overleftarrow{D^{3+}})^\nu \tilde{q}^\dagger_1 \bar{\sigma}_{\mu\nu} - \frac{1}{4}\tilde{q}^\dagger_1 \bar{\sigma}_\mu \not{D}^3 - \frac{1}{4}(\overleftarrow{D^{3+}})^\mu \tilde{q}^\dagger_1 \tag{b.30}$$

$$\{M^f_{Aq}\}^{A^{1\mu}q} = \frac{1}{2}(\overleftarrow{D^{3+}})^\nu \tilde{q}^\dagger_1 \bar{\sigma}_{\mu\nu} - i\frac{1}{4}\tilde{q}^\dagger_1 \bar{\sigma}_\mu \not{D}^3 - i\frac{1}{4}(\overleftarrow{D^{3+}})^\mu \tilde{q}^\dagger_1 \tag{b.31}$$

$$\{M^b_{Aq}\}^{A^{2\mu}q} = i\frac{1}{2}(\overleftarrow{D^{3+}})^\nu \tilde{q}^\dagger_1 \bar{\sigma}_{\mu\nu} - \frac{1}{4}\tilde{q}^\dagger_1 \bar{\sigma}_\mu \not{D}^3 - \frac{1}{4}(\overleftarrow{D^{3+}})^\mu \tilde{q}^\dagger_1 \tag{b.32}$$

$$\{M^b_{Aq}\}^{B^{1\nu\rho}q} = -i\frac{1}{2}\tilde{c}^3 \tilde{q}^\dagger_1 \bar{\sigma}^{\nu\rho} + \frac{1}{4}\tilde{B}^{3\nu\rho}\tilde{q}^\dagger_1 + i\tilde{B}^3_{+\mu}{}^\rho \bar{\sigma}^{\mu\nu}\tilde{q}^\dagger_1 \tag{b.33}$$

$$\{M^b_{Aq}\}^{B^{2\nu\rho}q} = \frac{1}{2}\tilde{c}^3 \tilde{q}^\dagger_1 \bar{\sigma}^{\nu\rho} - i\frac{1}{4}\tilde{B}^{3\nu\rho}\tilde{q}^\dagger_1 - \tilde{B}^3_{+\mu}{}^\rho \bar{\sigma}^{\mu\nu}\tilde{q}^\dagger_1 \tag{b.34}$$

$$\{M^f_{Aq}\}^{B^{1\nu\rho}q} = i\frac{1}{2}\tilde{c}^3 \tilde{q}^\dagger_1 \bar{\sigma}^{\nu\rho} + \frac{1}{4}\tilde{B}^{3\nu\rho}\tilde{q}^\dagger_1 - i\tilde{B}^3_{+\mu}{}^\rho \bar{\sigma}^{\mu\nu}\tilde{q}^\dagger_1 \tag{b.35}$$

$$\{M^f_{Aq}\}^{B^{2\nu\rho}q} = -\frac{1}{2}\tilde{c}^3\tilde{q}_1^\dagger\bar{\sigma}^{\nu\rho} - i\frac{1}{4}\tilde{B}^{3\nu\rho}\tilde{q}_1^\dagger + \tilde{B}^3_{+\mu}{}^{\rho}\bar{\sigma}^{\mu\nu}\tilde{q}_1^\dagger \tag{b.36}$$

$$\{M^b_{Aq}\}^{c^1q} = i\frac{1}{2}\tilde{B}^{3\mu\nu}_+\bar{\sigma}_{\mu\nu}\tilde{q}_1^\dagger - \frac{1}{4}\tilde{c}^3\tilde{q}_1^\dagger \tag{b.37}$$

$$\{M^b_{Aq}\}^{c^2q} = -\frac{1}{2}\tilde{B}^{3\mu\nu}_+\bar{\sigma}_{\mu\nu}\tilde{q}_1^\dagger + i\frac{1}{4}\tilde{c}^3\tilde{q}_1^\dagger \tag{b.38}$$

$$\{M^f_{Aq}\}^{c^1q} = -i\frac{1}{2}\tilde{B}^{3\mu\nu}_+\bar{\sigma}_{\mu\nu}\tilde{q}_1^\dagger - \frac{1}{4}\tilde{c}^3\tilde{q}_1^\dagger \tag{b.39}$$

$$\{M^f_{Aq}\}^{c^2q} = \frac{1}{2}\tilde{B}^{3\mu\nu}_+\bar{\sigma}_{\mu\nu}\tilde{q}_1^\dagger + i\frac{1}{4}\tilde{c}^3\tilde{q}_1^\dagger \tag{b.40}$$

For (b.29)∼(b.40), one can fine the relation

$$\begin{pmatrix} \{M^f_{Aq}\}^{Y^1q} \\ \{M^f_{Aq}\}^{Y^2q} \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \{M^b_{Aq}\}^{Y^1q} \\ \{M^b_{Aq}\}^{Y^2q} \end{pmatrix}. \tag{b.41}$$

Using above explicit matrix elements (b.20)∼(b.40), we can perform the path integral (b.12) directly, instead of neglecting non-kinetic off-diagonal part of $M^b(2)(M^f(2))$. Then we have a crucial obstacle from the difference between (b.23) and (b.24), while the obstacle from (b.29)∼(b.40) is resolved by the relation (b.41). This obstacle tells us that the contributions from (b.23) and (b.24) is not 1 and that the result (3.67) is effective up to order of square of $\tilde{q}_1$. (In fact this problem does not appear when we treat adjoint matter instead of fundamental matter. Thus we think that this problem comes from the choice of the representation of matter fields.) However the contributions from (b.23) and (b.24) becomes 1 in $\tilde{q}_1 \to 0$ limit after path integration. Thus we estimate that the contributions from (b.23) and (b.24) to be 1 in the case that the result $Z^t_{m,c,k}(2)$ (3.67) is topological. This is why it is enough to estimate the path integral with the indexes that the only kinetic terms in diagonal block are counted from the big matrices in section 3.

References

[1] C.Vafa and E.Witten,*A Strong coupling test of S-duality*, Nucl.Phys.B431(1994)3

[2] C.Lozano,*Duality in Topological Quantum Field Theories* , hep-th/9907123

[3] C.Montonen and D.Olive, Phys.Lett.B72(1977)117;
P.Goddard,J.Nyuts and D.Olive , Nucl.Phys.B125(1977)1

[4] S.Mukai,Inv.Math.77(1984)101;
L.Göttsche,Math.Ann.286(1990)193

[5] C.Vafa, *Instantons on D-branes*, Nucl.Phys.B463 (1996)435
M. Bershadsky, C. Vafa, V. Sadov, *D-branes and Topological Field Theories* Nucl.Phys.B463(1996)420
C.Vafa, *Gas of D-branes and Hagedorn Density of BPS States*, Nucl.Phys.B463(1996)415
M. Bershadsky, C. Vafa, V. Sadov, *D Strings on D Manifolds* Nucl.Phys.B463(1996)398

[6] Seungjoon Hyun, Jaemo Park, Jae-Suk Park *N=2 Supersymmetric QCD and Four Manifolds; (I) the Donaldson and the Seiberg-Witten Invariants* hep-th/9508162.
S. Hyun and J.-S.Park, $N = 2$ *Topological Yang-Mills Theories and Donaldson's Polynomials*, J.Geom.Phys. 20 (1996) 31-53

[7] S.K.Donaldson,*Polynomial invariants for smooth four-manifolds*, Topology 29(1990)257.
S.K.Donaldson and P.B.Kronheimer,*The geometry of four-manifolds*, (Oxford University Press,New York ,1990).
S.K.Donaldson,*An application of gauge theory to the topology of 4-manifolds*, J.Differential Geom.18(1983)269
S.K.Donaldson, *Connection, Cohomology, and intersection forms of 4-manifolds*, J.Differential Geom.26(1986)397.

[8] E.Witten,*Monopoles and four-manifolds*, Math.Research Lett.1(1994)769
E.Witten,*Supersymmetric Yang-Mills theory on a four manifolds*, J.Math.Phys.35(1994)5101.
E.Witten, *On S-duality in abelian gauge theory*, ,Selecta Mathematica 1(1995)383; hep-th/9505186.

[9] E.Witten, *Topological quantum field theory*, Comm.Math.Phys.117(1988)353
E.Witten, *Introduction to cohomological field theories*, Int.J.Mod.Phys.A.6(1991)2273.

[10] R.Dijkgraaf, J.Park, B.Schroers, *N=4 Supersymmetric Yang-Mills Theory on a Kähler Surface*,hep-th/9801066

[11] S.Hyun, Jaemo Park, Jae-Suk Park,*Topological QCD*, Nucl.Phys. B453 (1995) 199-224

[12] J.M.F.Labastida and M.Mariño, *Non-abelian monopoles on four-manifolds*, Nucl.Phys.B 448(1995)373,(hep-th/9504010)
M.Alvarez and J.M.F Labastida, *Topological matter in four dimensions*, Nucl.Phys.B 437(1995)356.
J.M.F.Labastida and M.Mariño,*A Topological Lagrangian for Monopoles on Four-manifolds* Phys. Lett. B351 (1995) 146

[13] R.Dijkgraaf and G.Moore, *Balanced topological field theories* Comm.Math.Phys.185(1997)411
J. M. F. Labastida, Carlos Lozano *Mathai-Quillen Formulation of Twisted $N = 4$ Supersymmetric Gauge Theories in Four Dimensions* ,Nucl.Phys. B502 (1997) 741

[14] M.Mariño, G.Moore, G.Peradze *Superconformal invariance and the geography of four-manifolds* Commun.Math.Phys. 205 (1999) 691-735

[15] M.Furuta, *Monopole equation and 11/8 conjecture*, preprint

[16] private discussion with Furuta Mikio

[17] A.Sako,*Reducible connections in massless topological QCD and 4-manifolds*, Nucl.Phys.B522(1998)373